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Fast Diffusion Equations with Caffarelli-Kohn-Nirenberg Weights: Regularity and Asymptotics

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A mi familia,
a Matteo,
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Contents

Introduction and summary of the main results

This Ph.D. thesis is devoted to the study of *diffusion equations*. A number of different natural phenomena can be described by means of such equations and they offer a rich mathematical theory. The simplest example appears to be the *Heat Equation*

$$u_t = \Delta u \,, \quad x \in \mathbb{R}^d \,, \quad t > 0 \,, \tag{HE}$$

introduced by Fourier in his seminal work, *Théorie analytique de la chaleur*, 1822, [1]. The function $u(t,x) : (0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ in (HE) describes the time evolution of the temperature in $\mathbb{R}^d$, given its initial distribution. Since then, (HE) has been of great relevance in both mathematics and other sciences. For instance, in [2] Einstein shows for the first time a link between the Brownian motion and the above (HE), opening the road to a better understanding of the theory of atoms and also the connections between stochastic processes and diffusion equations.

The mathematical theory of the (HE) is quite mature, see [3], and it will be our guide in understanding other models. We are interested in describing properties of solutions to (HE) such as existence, uniqueness, regularity and asymptotic behaviour. In the theory of diffusion equations, generally speaking, a few special explicit (or quasi-explicit) solutions show the typical behaviour of a large class of solutions. In the case of the Cauchy problem for (HE), posed on $\mathbb{R}^d$ with $d \geq 1$, the typical behaviour is encoded in the so called *fundamental solution*, which is called *Gaussian* and takes the form

$$\Gamma(t,x) = \frac{1}{(4\,\pi\,t)^{\frac{d}{2}}}\, e^{-\frac{|x|^2}{4\,t}} \,, \qquad x \in \mathbb{R}^d \,, \quad t > 0 \,. \tag{0.0.1}$$

The Gaussian has several remarkable properties: it is positive, radial, C^∞ regular in space and time and it *conserves mass* along the flow, namely $\|\Gamma(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|\Gamma(1,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)}$ for all $t > 0$. It is well-known that, for a large class of solution to (HE), the following *representation formula* holds

$$u(t,x) = u_0(\cdot) * \Gamma(t,\cdot) = \frac{1}{(4\,\pi\,t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u_0(y)\, e^{-\frac{|x-y|^2}{4\,t}} \,\mathrm{d}y \,. \tag{0.0.2}$$

By the properties of the convolution, it is clear that a large class of solutions inherit some of the aforementioned regularity properties of the Gaussian, as well as its qualitative behaviour for large times. We also remark that for the (HE) the *infinite speed of propagation* holds, namely the fact that nonnegative compactly supported initial data u_0 produce positive solutions $u(t,x) > 0$ for any $x \in \mathbb{R}^d$ and any $t > 0$.

However the theory of diffusion equations offers much more examples than the one above! Needless to say, not all the available models enjoy the properties of the above (HE) and a wide variety of phenomena may occur. A quite general form of nonlinear diffusion equation is

$$\partial_t H(t,x,u) = \sum_{i=1}^{d} \partial_{x_i} \left(A_i(t,x,u,\nabla u) \right) + B(t,x,u,\nabla u)\,,$$

under suitable conditions on H, A_i and B, see [4]. A theory for such equations, in the whole generality, has been investigated in the last few decades. However, the great variability within all nonlinear equations of that form precludes a unified approach of this study. A family of equations for which much is known is the *PME/FDE* given by the following simple model

$$u_t = \nabla \cdot (D(u)\,\nabla u) = \Delta u^m\,, \qquad m > 0\,, \tag{PME/FDE}$$

where $D(u) = m\,u^{m-1}$ is called the *diffusion coefficient*. We point out that for $m = 1$, the (PME/FDE) is exactly the (HE), when $m > 1$ the above is called *porous medium equation* while when $0 < m < 1$ it is called *fast diffusion equation*. The (PME/FDE) is an example of *non linear parabolic equation* which does not share the behaviour of the (HE). Here are some examples. In the case $m > 1$ solutions have *finite speed of propagation*, namely the fact that compactly supported data produce compactly supported solutions. While when $0 < m < 1$ positive solutions may vanish after finite time, a fact in contradiction with the conservation of mass. When $m > \frac{d-2}{d}$ the whole family (PME/FDE) does admit the existence of a fundamental solution (which is often called the *Barenblatt solution*) but no representation formula is available.

The typical techniques used in the study of the linear (HE) cannot be adapted to this nonlinear setting. For instance, in the case of (PME/FDE), $m > 1$, the fundamental solution given by

$$\mathcal{U}(t,x) = t^{-\frac{d}{2+d(m-1)}} \left(C - \kappa\,|x|^2\, t^{\frac{2}{2+d(m-1)}} \right)_+^{\frac{1}{m-1}}\,,$$

is *not* classical. Hence new concepts of weak solutions were introduced in the last century in order to solve this issue. This requires advanced analytic techniques such as Sobolev spaces, distributions, weak derivatives, etc. To prove that generalized solutions to nonlinear parabolic (or elliptic) equations are in fact regular (or classical) was a very hot topic in the last century, since it can be considered as the parabolic version of the 19^{th} Hilbert problem.

This thesis is divided in two parts. Part I contains 3 chapters and is dedicated to the study of the regularity properties of local solutions to the *weighted fast diffusion equation*. Part II, which also contains 3 chapters, is focused on other issues. Chapters 4 and 5 are dedicated to the study of the long time behaviour of solutions to the weighted fast diffusion equation and the *fast p-Laplace evolution equation*. Lastly, in Chapter 6 we focus our attention on the stability of the *Gagliardo-Nirenberg* inequalities.

Part I and Chapter 4 of this thesis are dedicated to the study of the following model, which we call *weighted fast diffusion equation*

$$u_t = |x|^{\gamma} \nabla \cdot \left(|x|^{-\beta} \nabla u^m \right)\,, \tag{WFDE}$$

with $d \geq 3$ and $m \in (0,1)$. We will always consider the following range of parameters

$$\gamma < d \qquad \text{and} \qquad \gamma - 2 < \beta \leq \frac{d-2}{d}\gamma\,. \tag{0.0.3}$$

This range is optimal for the validity of the following family of Caffarelli-Kohn-Nirenberg inequalities [5]

$$\|w\|_{\mathrm{L}^{2p}_{\gamma}(\mathbb{R}^d)} \le \mathcal{C}_{\gamma,\beta,p} \|\nabla w\|^{\xi}_{\mathrm{L}^2_{\beta}(\mathbb{R}^d)} \|w\|^{1-\xi}_{\mathrm{L}^{p+1}_{\gamma}(\mathbb{R}^d)} \qquad \text{for any} \qquad w \in C^\infty_c(\mathbb{R}^d)\,, \tag{0.0.4}$$

where

$$\|w\|_{\mathrm{L}^q_\gamma(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q \frac{\mathrm{d}x}{|x|^\gamma}\right)^{\frac{1}{q}}, \quad p \in \left(1, \frac{d-\gamma}{d-2-\beta}\right), \quad \xi = \frac{(d-\gamma)(p-1)}{p(d+2+\beta-2\gamma-p(d-2-\beta))}\,,$$

and γ, β are as in (0.0.3).

The nonlinear equation (WFDE) was introduced in the 80s by Kamin and Rosenau [6, 7, 8], to model singular/degenerate diffusion in inhomogeneous media. Very recently (WFDE) has been investigated for its in connection with the issue of symmetry/symmetry-breaking of optimal function in the family of Caffarelli-Kohn-Nirenberg inequalities of (0.0.4), see [9].

In the case $\gamma = \beta = 0$, (WFDE) is exactly the aforementioned fast diffusion equation whose weak solutions enjoy C^∞ regularity. However this is not always the case for the (WFDE). Indeed, when $\frac{d-2-\beta}{d-\gamma} < m < 1$ the (WFDE) admits a family of fundamental solutions, called Barenblatt profiles, given by

$$\mathfrak{B}(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0 \frac{t^{(2+\beta-\gamma)\vartheta}}{M^{(2+\beta-\gamma)\vartheta(1-m)}} + b_1|x|^{2+\beta-\gamma}\right]^{\frac{1}{1-m}}}\,, \tag{B-WFDE}$$

where M is the “mass”, b_0, b_1 are parameters which depend on d, m, γ, β and $\vartheta^{-1} = 2+\beta-\gamma-(d-\gamma)(1-m)$. It is clear from (B-WFDE) that the regularity at the origin of $\mathfrak{B}$ depends on the value of $2+\beta-\gamma$.

In Part I we study a priori estimates for a class of non-negative local weak solution to the (WFDE), with $0<m<1$ posed on cylinders of $(0,T)\times\mathbb{R}^d$. The weights $|x|^\gamma$ and $|x|^{-\beta}$ can be both degenerate and singular and need not satisfy some typical assumption for these kind of problems (for instance the do not belong to the class $\mathcal{A}_2$, see Part I of this thesis). The weights are not translation invariant and this causes a number of extra difficulties and a variety of scenarios: for instance, the scaling properties of the equation change when considering the problem around the origin or far from it. In Chapters 1 and 2 we prove quantitative - with computable constants - upper and lower estimates for local weak solutions, focusing our attention where a change of geometry appears. In Chapter 3 we show how the above estimates combine into forms of Harnack inequalities of forward, backward and elliptic type. As a consequence, we obtain Hölder continuity of the solutions, with an quantitative (even if non-optimal) exponent. We stress the fact that, due to the explicit solution (B-WFDE), not much more can be said in such generality. The proof of the positivity estimates requires a new method and represents one of the main technical novelties of this thesis. In the linear case, $m=1$, we also prove quantitative estimates, recovering known results in some cases and extending such results to a wider class of weights.

The results presented in Part I are contained in

M. Bonforte, N. Simonov. *Quantitative a priori estimates for fast diffusion equations with Caffarelli-Kohn-Nirenberg weights. Harnack inequalities and Hölder continuity*, Advances in Mathematics, **345** 2019, 1075–1161

In Chapter 4 of Part II we study the *asymptotic behaviour* of the Cauchy problem for a particular class of solutions to (WFDE). It is know, at least in the case $\gamma=\beta=0$, that the Barenblatt profile (WFDE) represent a qualitative model for the long time behaviour for L^1 initial data. However, several scenarios are available, and not all the solutions approach $\mathfrak{B}(t,x)$ in the same way.

Our main result in this chapter is the following: we characterize the largest class of initial data for which the so-called Global Harnack Principle holds (a global lower and upper bound in terms of suitable Barenblatt solutions). We call this class $\mathcal{X}$: is a suitable subspace of L^1 whose functions decay at infinity (namely for large $|x|$) a in a precise way, characterized by a sharp integral behaviour.

As a consequence of the GHP, we prove convergence in *uniform relative error*, namely that

$$\lim_{t\to\infty}\left\|\frac{u(t,x)-\mathfrak{B}(t,x)}{\mathfrak{B}(t,x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}=0\,.$$

In the case with no weights (namely $\gamma=\beta=0$) and for data in $\mathcal{X}$ we give (almost) sharp rates of convergence in the uniform relative error, in the radial case we give sharp rates. We prove (non optimal) rates of convergence in the WFDE case. We also give semi-explicit examples of solutions which belongs to $\mathcal{X}^c$ and have a different behaviour: for such solutions there is no uniform convergence of the relative error and no GHP holds. However, under suitable assumptions, a *generalized* version of the GHP holds.

Lastly for the WFDE we prove that $\mathcal{X}$ and $\mathcal{X}^c$ are stable under the flow, namely initial data in $\mathcal{X}$ (or in $\mathcal{X}^c$) produce solutions which stay in $\mathcal{X}$ (in $\mathcal{X}^c$ respectively) for all times. We stress on the fact that our results are new even in the standard framework of $\gamma=\beta=0$ and $\frac{d-2}{d}<m<1$.

Chapter 5 of Part II is dedicated to the study of the following fast p-Laplace equation (PLE):

$$u_t(t,x)=\Delta_p u(t,x)=\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right). \tag{PLE}$$

This model represent a different kind of diffusion with respect to the one considered in (PME/FDE). The (PLE) belongs to the class of *gradient dependent diffusion equations* and its diffusion coefficient is $D(\nabla u)=|\nabla u|^{p-2}$. We recall that for $p=2$ we recover again (HE). The (PLE) model has been used to describe the filtration through a porous medium for a non-newtonian fluid, see [10], and, for example, in image processing, see [11].

In Chapter 5 we investigate the Cauchy problem, for $d\ge 3$, in the range of parameters $\frac{d+1}{2d}<p<2$, this regime is generally called *fast diffusion*. We prove sharp, global lower bounds for solution to the Cauchy Problem with non-negative initial data $u_0\in\mathrm{L}^1(\mathbb{R}^d)$. In this case, a fundamental solution is also available. It is called the Barenblatt profile for the p-Laplace evolution equation and it is given by:

$$\mathcal{B}(t,x)=t^{\frac{1}{2-p}}\left[b_2\frac{t^{\frac{\beta p}{p-1}}}{M^{(2-p)\frac{\beta p}{p-1}}}+b_3\,|x|^{\frac{p}{p-1}}\right]^{-\frac{p-1}{2-p}}, \tag{B-PLE}$$

where $\beta^{-1}=p+d(p-2)$, b_2,b_3 depend on d,p and M is the mass of $\mathcal{B}(t,x)$. It was already known that the Barenblatt profile $\mathcal{B}(t,x)$ describes the asymptotic behaviour of solutions to (PLE) of mass M, however, as for (WFDE), many scenarios can occur. We characterize the class of initial data for which the Global Harnack Principle holds, it is a generalization of the aforementioned class $\mathcal{X}$. As a consequence of the GHP we prove convergence of uniform relative error. Finally, exploiting a connection among (PLE) and (WFDE), we give sharp quantitative behaviour of the gradient ∇u for a class of radial solutions. We remark that most of our results are new.

In Chapter 6 we turn our attention to the issue of stability of functional inequalities. In the case $\gamma=\beta=0$, (0.0.4) is nothing else than a particular class of Gagliardo-Nirenberg inequalities, which can be also written in the equivalent (but not scaling invariant) form

$$\delta[f]:=(p-1)^2\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+4\,\frac{d-p\,(d-2)}{p+1}\,\|f\|^{p+1}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}-2\,\mathcal{K}_{\mathrm{GN}}\,\|f\|^{2p\,\frac{d\,(p-1)-2\,(p+1)}{d\,(p-1)-4\,p}}_{\mathrm{L}^{2p}(\mathbb{R}^d)}\ge 0\,. \tag{0.0.5}$$

We will call δ the *deficit functional*. In [12], Del-Pino and Dolbeault were able to compute the best constants for the family (0.0.4) in the non-weighted case $\gamma = \beta = 0$. In the same paper they also established a deep connection between the deficit functional δ and the (PME/FDE). Indeed, $\delta \geq 0$ is nothing else than the *entropy-entropy production* inequality for the (PME/FDE) flow.

Since the seminal paper of Bianchi-Egnell [13], several stability results have been obtained, see [14, 15, 16, 17]. However few results provide quantitative estimates with explicit constants. In Chapter 6 we develop a new strategy for studying the stability of the aforementioned class of Gagliardo-Nirenberg inequalities, in which the fast diffusion flow is used as a tool. The main novelty is that, using the estimates proven in Chapters 1,2,3 and 4, we can quantify all steps of our constructive proof. As a consequence, we are able to establish explicit lower bounds for the deficit δ in terms of the relative entropy and of the Fisher information associated with the fast diffusion flow.

The results presented in Part II are contained in

M. Bonforte, N. Simonov. *The Global Harnack Principle for the Weighted Fast Diffusion Equation with Caffarelli-Kohn-Nirenberg weights*, Preprint, 2019

and in

M. Bonforte, N. Simonov, D. Stan. *Global Positivity Estimates for the p-Laplace Evolution Equation*, Final stage of preparation, 2019

and in

M. Bonforte, J. Dolbeault, B. Nazaret, N. Simonov. *From the Fast Diffusion Flow to Stability in Gagliardo-Nirenberg Inequalities*, Preprint, 2020

Introducción y presentación de los resultados

La presente tesis doctoral está dedicada al estudio de las ecuaciones de difusión. Tales ecuaciones emergen de modo natural en el estudio de una amplia gama de problemas físicos y, asímismo, tienen un interés matemático propio de gran amplitud y riqueza.

El ejemplo más simple es la Ecuación del Calor (Heat Equation):

$$u_t = \Delta u\,, \quad x \in \mathbb{R}^d\,, \quad t > 0\,, \tag{HE}$$

introducida por Fourier en su trabajo clásico, *Théorie analytique de la chaleur*, 1822, [1]. La función $u(t,x) : (0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ in (HE) describe la evolución temporal de la temperatura en $\mathbb{R}^d$, una vez la distribución inicial es conocida. Desde entonces, la (HE) ha tenido mucha relevancia en las matemáticas y en otras ciencias. Por ejemplo, en [2] Einstein mostró la relación entre el movimiento browniano y dicha ecuación, iniciando así el camino hacia una comprensión más profunda de los átomos, así como de la conexión entre procesos estocásticos y ecuaciones de difusión.

Actualmente, el conocimiento de la (HE) es muy profundo, véase [3], por lo que puede usarse como guía para el estudio de otros modelos. Nuestro interés principal es el estudio de las propiedades de las soluciones de la (HE), como la existencia, unicidad, regularidad y comportamiento asintótico. En el campo de las ecuaciones de difusión, grosso modo, se puede decir que unas pocas soluciones explícitas son suficientes para entender el comportamiento típico de una clase más amplia. En el caso del problema de Cauchy para la (HE), planteado en $\mathbb{R}^d$ con $d \geq 1$, el comportamiento típico está dado por la así llamada *solución fundamental*, que se llama *Núcleo gaussiano* y está dado por:

$$\Gamma(t,x) = \frac{1}{(4\,\pi\,t)^{\frac{d}{2}}}\,e^{-\frac{|x|^2}{4\,t}}\,, \qquad x \in \mathbb{R}^d\,, \quad t > 0\,. \tag{0.0.6}$$

Este núcleo tiene varias propiedades notables: es positivo, radial, es suave en todas sus variables y conserva la masa a lo largo del flujo; es decir: $\|\Gamma(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|\Gamma(1,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)}$ para todo $t > 0$. Es bien sabido que, para una amplia clase de soluciones de la (HE), se tiene la siguiente *fórmula de representación*:

$$u(t,x) = u_0(\cdot) * \Gamma(t,\cdot) = \frac{1}{(4\,\pi\,t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u_0(y)\,e^{-\frac{|x-y|^2}{4\,t}}\,\mathrm{d}y\,. \tag{0.0.7}$$

Debido a las propiedades de la convolución, es evidente que muchas soluciones heredan las propiedades de regularidad arriba mencionadas, así como el comportamiento cualitativo para tiempos grandes. También mencionamos que las soluciones de la (HE) presentan *velocidad de propagación infinita*; es decir: un dato inicial no negativo de soporte compacto da lugar a soluciones estrictamente positivas para todo tiempo $t > 0$.

No obstante, el campo de las ecuaciones de difusión ofrece muchos más ejemplos que éste! Huelga decir que no todos los modelos conocidos disfrutan de las propiedades típicas de la (HE) y que, de hecho, pueden producirse una amplia variedad de fenómenos en el estudio de los mismos. Una formulación muy general de una ecuación no lineal difusiva es:

$$\partial_t H(t,x,u) = \sum_{i=1}^{d} \partial_{x_i} \left(A_i(t,x,u,\nabla u) \right) + B(t,x,u,\nabla u) \,,$$

bajo condiciones adecuadas para H, A_i y B, véase [4]. Durante las últimas décadas, ha habido un esfuerzo intenso por desarrollar una teoría lo más general posible de este tipo de ecuaciones. Sin embargo, debido a la enorme variedad entre esa familia de ecuaciones, es imposible hallar un enfoque unificado a su estudio. Una familia de ecuaciones para las cuales existe una teoría muy desarrollada es la de las *PME/FDE*, dadas por:

$$u_t = \nabla \cdot (D(u)\, \nabla u) = \Delta u^m \,, \qquad m > 0 \,, \tag{PME/FDE}$$

donde $D(u) = m\, u^{m-1}$ es el *coeficiente de difusión.* Para $m = 1$, la (PME/FDE) se reduce a la (HE), mientras que para $m > 1$ o $m < 1$ la ecuación se denomina, respectivamente, *Porous Medium Equation* (Ecuación de Medios Porosos) y *Fast Difussion Equation* (Ecuación de Difusión Rápida). Esta familia de ecuaciones ofrece un ejemplo de ecuación no lineal parabólica cuyo comportamiento difiere del de la (HE). Por ejemplo, para $m > 1$ las soluciones presentan velocidad finita de propagación, lo que significa que los datos iniciales de soporte comopacto dan lugar a soluciones de soporte compacto, para $0 < m < 1$ las soluciones no negativas se extinguen en tiempo finito, un hecho que entra en contradicción con la posible conservación de la masa, y para $m > (d-2)/d$ todas las ecuaciones de la familia poseen una solución fundamental (la solución de Barenblatt), pero sus soluciones no admiten una fórmula de representación.

Las técnicas habituales empleadas en el estudio de la (HE) no pueden extrapolarse al contexto de esta familia de ecuaciones no lineales. Por ejemplo, en el caso $m > 1$, la solución fundamental está dada por:

$$\mathcal{U}(t,x) = t^{-\frac{d}{2+d(m-1)}} \left(C - \kappa\, |x|^2\, t^{\frac{2}{2+d(m-1)}} \right)_+^{\frac{1}{m-1}} \,,$$

que no es *clásica.* Así pues, en el último siglo se hizo necesaria la introducción de un nuevo concepto, el de solución débil, adecuada para este tipo de problemas. Esto requirió de una serie de técnicas analíticas avanzadas, como son los espacios de Sobolev, la teoría de distribuciones y las derivadas débiles. La regularidad de las soluciones generalizadas a las ecuaciones parabólicas (elípticas) no linales fue uno de los problemas que más atención recibieron durante el último siglo, puesto que se pueden considerar como una versión parabólica del decimonoveno problema de Hilbert.

La presente tesis está dividida en dos partes. La primera parte contiene tres capítulos y está dedicada al estudio de las propiedades de regularidad de soluciones locales de la *ecuación de difusión rápida con pesos* (WFDE). La segunda parte, consistente también en tres capítulos, se centra en otros aspectos. Los capítulos cuatro y cinco estudian el comportamiento para tiempos grandes de las soluciones a la ecuación de difusión rápida con pesos y de las de la *ecuación de evolución rápida para el p-laplaciano.* Por último, en el capítulo seis nos centramos en el estudio de la estabilidad de las *desigualdades de Gagliardo-Nirenberg.*

La primera parte y el capítulo cuatro de la tesis se centran en el estudio del siguiente modelo, al que llamaremos ecuación de difusión rápida con pesos (WFDE):

$$u_t = |x|^{\gamma} \nabla \cdot \left(|x|^{-\beta} \nabla u^m \right) \,, \tag{WFDE}$$

con $d \geq 3$ y $m \in (0,1)$. El rango de parámetros a considerar será siempre:

$$\gamma < d \qquad \text{and} \qquad \gamma - 2 < \beta \leq \frac{d-2}{d}\gamma\,. \tag{0.0.8}$$

Dicho rango es óptimo respecto a la validez de la siguiente familia de desigualdades de Caffarelli-Kohn-Nirenberg, véase [5]:

$$\|w\|_{\mathrm{L}^{2p}_\gamma(\mathbb{R}^d)} \leq \mathcal{C}_{\gamma,\beta,p}\,\|\nabla w\|^{\xi}_{\mathrm{L}^2_\beta(\mathbb{R}^d)}\,\|w\|^{1-\xi}_{\mathrm{L}^{p+1}_\gamma(\mathbb{R}^d)} \qquad \text{for any} \qquad w \in C^\infty_c(\mathbb{R}^d)\,, \tag{0.0.9}$$

donde

$$\|w\|_{\mathrm{L}^q_\gamma(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q \,\frac{\mathrm{d}x}{|x|^\gamma}\right)^{\frac{1}{q}}\,, \quad p \in \left(1, \frac{d-\gamma}{d-2-\beta}\right)\,, \quad \xi = \frac{(d-\gamma)(p-1)}{p(d+2+\beta-2\gamma-p(d-2-\beta))}\,,$$

y γ, β definidos como en (0.0.8).

La ecuación no lineal (WFDE) fue introducida a mediados de los años ochenta por Kamin y Rosenau [6, 7, 8] para modelizar procesos de difusión singular o degenerada en medios no homogéneos. Recientemente la ecuación (WFDE) ha recibido atención por su conexión con el problema de la ruptura de simetría de funciones óptimas en la familia de desigualdades Caffarelli-Kohn-Nirenberg de (0.0.9), véase [9].

En el caso $\gamma = \beta = 0$, (WFDE) coincide con la ecuación de difusión rápida mencionada arriba, cuyas soluciones débiles poseen regularidad C^∞ . Sin embargo, esto no siempre es así en el caso de la (WFDE). De hecho, cuando $\frac{d-2-\beta}{d-\gamma} < m < 1$ la (WFDE) admite una familia de soluciones fundamentales, llamados perfiles de Barenblatt, dados por:

$$\mathfrak{B}(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0 \frac{t^{(2+\beta-\gamma)\vartheta}}{M^{(2+\beta-\gamma)\vartheta(1-m)}} + b_1|x|^{2+\beta-\gamma}\right]^{\frac{1}{1-m}}}\,, \tag{B-WFDE}$$

donde M es la "masa", b_0, b_1 son parámetros que dependen de d, m, γ, β y $\vartheta^{-1} = 2+\beta-\gamma-(d-\gamma)(1-m)$. La regularidad de $\mathfrak{B}$ en el origen dependen, como se desprende de su definición (B-WFDE) , del valor de $2+\beta-\gamma$.

En la primera parte, estudiamos estimaciones a priori para una clase de soluciones débiles no negativas y locales de la (WFDE) propuesta en cilindros $(0,T) \times \mathbb{R}^d$ y con $0 < m < 1$. Los pesos $|x|^\gamma$ y $|x|^{-\beta}$ pueden ser ambos degenerados y singulares y no tienen por qué satisfacer las asunciones típicas que suelen hacerse en este tipo de problemas (por ejemplo, no pertenecen a la clase $\mathcal{A}_2$, véase la primera parte de esta tesis). Los pesos no poseen invariancia traslacional, lo cual causa varias dificultades adicionales y una variedad de posibles escenarios: las propiedades de escala de la ecuación se ven modificadas cuando el problema es considerado cerca del origen o lejos de él. En los capítulos 1 y 2 probamos estimaciones cuantitativas superiores e inferiores -con constantes calculables- para soluciones débiles, centrándonos en situaciones en las que hay un cambio de geometría. En el tercer capítulo mostramos como dichas estimaciones se combinan para dar lugar a desigualdades de tipo Harnack de distintos tipos. Como consecuencia, derivamos la continuidad Hölder de las soluciones y damos el correspondiente exponente (incluso si no es óptimo). La prueba de las estimaciones de positividad requiere un nuevo método y representa una de las novedades técnicas más relevantes de esta tesis. En el caso lineal, $m = 1$, también probamos estimaciones cuantitativas que recuperan resultados conocidos en algunos casos y que los extienden a una clase de pesos más amplia. Los resultados de la primera parte de la tesis se encuentra en:

M. Bonforte, N. Simonov. *Quantitative a priori estimates for fast diffusion equations with Caffarelli-Kohn-Nirenberg weights. Harnack inequalities and Hölder continuity*, Advances in Mathematics, **345** 2019, 1075–1161

En el capítulo 4 de la segunda parte estudiamos el comportamiento asintótico en el problema de Cauchy para una clase particular de soluciones a la (WFDE). Es sabido, al menos en el caso $\gamma = \beta = 0$, que el perfil de Barenblatt $\mathfrak{B}$ representa un modelo cualitativo para el comportamiento en tiempos grandes para datos iniciales en L^1. No obstante, se pueden presentar distintos escenarios, y no todas las soluciones se acercan a $\mathfrak{B}$ de la misma manera.

El resultado principal de este capítulo es la caracterización de la clase más amplia de datos iniciales para los cuales el Principio Global de Harnack (GHP) se satisface (una cota superior y una inferior en términos de soluciones de Barenblatt adecuadas). Llamamos $\mathcal{X}$ a esta clase, que es un subespacio adecuado de L^1 cuyas funciones miembro decaen en el infinito de una manera particular.

Como consecuencia del GHP, probamos convergencia *uniforme en error relativo*; es decir

$$\lim_{t\to\infty}\left\|\frac{u(t,x)-\mathfrak{B}(t,x)}{\mathfrak{B}(t,x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0\,.$$

En el caso sin pesos (es decir $\gamma = \beta = 0$) y para datos iniciales en $\mathcal{X}$ damos tasas (casi) optimas para la converfence uniforme en error relativo, en el caso radial las tasas son optimas. Para la (WFDE) damos tasas de convergencia no optimas. Costruimos ejemplos semi explicitos de soluciones cuyo dato inicial pertenece a $\mathcal{X}^c$ y para estas soluciones el comportamiento para tiempo largos es distinto: no hay convergencia uniforme en error relativo y el GHP no se cumple. Sin embargo, bajo ciertas condiciones, se cumple una version *generalizada* del GHP.

Por último probamos que los conjuntos $\mathcal{X}$ y $\mathcal{X}^c$ son estables a lo largo del flujo; es decir, datos iniciales en $\mathcal{X}$ (o en $\mathcal{X}^c$) producen soluciones que estaán en $\mathcal{X}$ (en $\mathcal{X}^c$ respectivamente) para todo tiempo $t > 0$. Estos resúltados son novedosos también en caso de $\gamma = \beta = 0$ y $\frac{d-2}{d} < m < 1$.

El Capítulo 5 de la segunda parte se centra en el estudio del siguiente modelo de difusión rapida para el p-Laplaciano (PLE)

$$u_t(t,x) = \Delta_p u(t,x) = \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right). \tag{PLE}$$

Este último modelo rapresenta una clase distinta de ecuaciones de difusión. La (PLE) es una ecuación cuyo coeficiente de difusión $D(\nabla u) = |\nabla u|^{p-2}$ depende del gradiente de la solución. Para $p = 2$ la (PLE) se reduce a (HE).

En el Capítulo 5 estudiamos el comportamiento asintótico en el problema de Cauchy, con $d \geq 3$ y $\frac{d+1}{2d} < p < 2$. El mencionado rango de parametros es llamado *difusión rapida*. En este capitulo consideramos datos iniciales $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ y probamos estimaciones cuantitativas globles (es decir sobre todo $\mathbb{R}^d$) superiores y inferiores. También en este caso existe una solución fundamental, llamada perfíl de Barenblatt para la ecuación de difusión rapida del p-Laplaciano, dada por:

$$\mathcal{B}(t,x) = t^{\frac{1}{2-p}}\left[b_2\frac{t^{\frac{\beta p}{p-1}}}{M^{(2-p)\frac{\beta p}{p-1}}} + b_3\,|x|^{\frac{p}{p-1}}\right]^{-\frac{p-1}{2-p}}, \tag{B-PLE}$$

donde $\beta^{-1} = p + d(p-2)$, b_2, b_3 dependen de d, p y M es la "masa" de $\mathcal{B}(t,x)$. Es sabido que el perfil de Barenblatt $\mathcal{B}(t,x)$ desribe el comportamiento asintótico de las soluciones de (PLE), sin embargo, como en el caso de (WFDE), se pueden presentar distintos escenarios. En este capítulo caracterizamos la clase más amplia de soluciones para las cuales se cumple es GHP y la convergencia uniforme del error relativo: es una generalización de la ya mencionada clase $\mathcal{X}$. Por último, utilizamos una conexion entre la ecuación (PLE) y la (WFDE), para probar estimaciones superiores cuantitativas sobre el gradiente ∇u de soluciones radial de la (PLE).

El Capítulo 6 se centra en el estudio de la estabilidad de desigualdades funcionales. En el caso $\gamma = \beta = 0$, (0.0.9) es una clase particular de desigualdades de Gagliardo-Nirenberg, que se pueden escribir de forma equivalente como

$$\delta[f] := (p-1)^2 \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + 4 \, \frac{d - p\,(d-2)}{p+1} \, \|f\|^{p+1}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} - 2\,\mathcal{K}_{\mathrm{GN}} \, \|f\|^{2p \, \frac{d\,(p-1) - 2\,(p+1)}{d\,(p-1) - 4\,p}}_{\mathrm{L}^{2p}(\mathbb{R}^d)} \ge 0 \,. \quad (0.0.10)$$

El funcional δ es llamado *funcional de deficit* En [12], Del-Pino y Dolbeault lograron calcular la mejor constante para la clase (0.0.9) en el caso de $\gamma = \beta = 0$. En el mismo artículo logran establecer una conexión profunda entre el functional de deficit δ y la ecuación (PME/FDE). En efecto, $\delta \ge 0$ es la desigualdad de *entropía-producción de entropía* a lo largo del flujo (PME/FDE).

Desde el artículo clásico de Bianchi-Egnell [13], se han probado varios resultados de estabilidad, por ejemplo véase [14, 15, 16, 17]. Sin embargo, pocos resultados son verdaderamente cuantitativos, es decir: no se conocen muchas estimaciones con constantes explicitas. En el Capítulo 6 logramos desarrolar una nueva estrategia pare el estudio de la estabilidad de las mencionadas desigualdades de Gagliardo-Nirenberg. Utilizamos el flujo de difusión rapida como una herramienta y, empleamos las estimaciones probadas en los capitulos 1,2,3 y 4 para cuantificar cada paso de nuestra prueba constructiva. Como consecuencia, conseguimos establecer una acota inferior del funcional δ en términos de la entropía relativa y de la información de Fisher asociadas con el flujo de la difusión rápida.

Los resultados de la segunda parte están contenidos en

M. Bonforte, N. Simonov. *The Global Harnack Principle for the Weighted Fast Diffusion Equation with Caffarelli-Kohn-Nirenberg weights*, Preprint, 2019

y

M. Bonforte, N. Simonov, D. Stan. *Global Positivity Estimates for the p-Laplace Evolution Equation*, Final stage of preparation, 2019

y

M. Bonforte, J. Dolbeault, B. Nazaret, N. Simonov. *From the Fast Diffusion Flow to Stability in Gagliardo-Nirenberg Inequalities*, Preprint, 2020

Part I

A Priori Estimates for Fast Diffusion Equations with Caffarelli-Kohn-Nirenberg weights

Introduction to Part I

We investigate quantitative a priori estimates and regularity properties of nonnegative solutions to nonlinear singular diffusion equations with weights, possibly degenerate or singular, whose prototype is given by the following Weighted Fast Diffusion Equation

$$u_t = |x|^{\gamma}\nabla\cdot\left(|x|^{-\beta}\nabla u^m\right) \tag{WFDE}$$

posed on a domain of $(0,+\infty)\times\mathbb{R}^N$, with $N\geq 3$ and $m\in(0,1)$. We will always consider the following range of parameters, see also Figure 2 on page 33:

$$\gamma<N \qquad \text{and} \qquad \gamma-2<\beta\leq\frac{N-2}{N}\gamma\,. \tag{0.0.11}$$

This range of parameters is optimal for the validity of a family of the so-called Caffarelli-Kohn-Nirenberg inequalities [5]: let $r^*:=2(N-\gamma)/(N-(2+\beta))$,

$$\left(\int_{\mathbb{R}^N}|f|^{r^*}|x|^{-\gamma}\,\mathrm{d}x\right)^{1/r^*}\leq\overline{S}_{\gamma,\beta}\left(\int_{\mathbb{R}^N}|\nabla f|^2|x|^{-\beta}\,\mathrm{d}x\right)^{1/2}, \tag{0.0.12}$$

see Subsection 0.0.3 for more details; these inequalities are deeply connected with the above WFDE, in its evolutionary or stationary version, see for instance [18, 19, 20, 21, 9, 22, 23]; some further connection will be discussed and explored in the following chapters.

A priori estimates are the cornerstone of the theory of nonlinear partial differential equations. The main purpose of Part I is to prove precise quantitative local upper and lower bounds which combine into different forms of Harnack inequalities; as a consequence we also prove interior Hölder continuity for solutions to this class of equations with a (small) quantified exponent: the optimal Hölder exponent is not known. Indeed, in the case of the Cauchy problem some explicit (Barenblatt-type) solutions are known to be only Hölder continuous at $x=0$, as we shall discuss later, see also [18, 19].

The weights that we consider are not translation invariant and this causes a number of extra difficulties and a variety of scenarios that we explain in Subsection 0.0.1. Roughly speaking, the scaling properties of the equation change from $R^{2+\beta-\gamma}$ to (a multiple of) R^2, when we are considering the problem around the origin or far from it, respectively. We focus on the cases in which the change of geometry plays a role: in the other cases, the results are essentially the same as the classical ones, cf. [24, 25].

Our quantitative interior estimates are formulated for nonnegative local strong solutions, defined in Subsection 0.0.1. A number of interesting problems fall into our setting, for instance, the Cauchy problem on $\mathbb{R}^N$, problems on Euclidean domains with different boundary condition (Dirichlet, Neumann, Robin, etc.), as well as the so-called "large solutions" (which tend to $+\infty$ at the boundary of the domain). Moreover, our estimates can be extended to a wider class of solutions, through lengthy but standard approximations. We prove analogous results also in the linear case $m = 1$, as we shall discuss below.

The above nonlinear equation was introduced in the 80s by Kamin and Rosenau [6, 7, 8], to model heat propagation -or more generally singular/degenerate diffusion- in inhomogeneous media; the parabolic problem has been studied by many authors since then, mostly in the case $m \geq 1$ and with only one weight [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47].

In the non-weighted case $\gamma = \beta = 0$, the WFDE becomes the standard Fast Diffusion Equation (FDE) which has been intensively studied in the recent years by many authors: it is hopeless to give here a complete bibliography, hence we refer to the monographs [48, 4] and [49, 25] for a complete account, as well as for the physical relevance of the model. We just remark that our results hold also in the non-weighted case, and we recover the previous results with a different proof.

More recently, WFDE has been investigated for its deep relation with the so-called Caffarelli-Kohn-Nirenberg inequalities, [5]; in particular, the intriguing issue of symmetry/symmetry breaking, has attracted the attention of many prominent researchers, [18, 19, 50, 20, 21, 9, 22, 23, 51]. The study of such problem partly relies on the study of the Cauchy problem for the WFDE on $\mathbb{R}^N$ for which the regularity estimates of this work are fundamental and were not present in the literature: sometimes an extra hypothesis had to be added to fix this issue. This happens for instance in [18, 19], where the sharp asymptotic behaviour of solutions to the Cauchy problem for WFDE is studied: the regularity estimates proven here are indeed essential to ensure the validity of those results in full generality.

Lately, new connections between weighted parabolic equations and nonlinear diffusions on Riemannian manifolds were explored in [52, 53, 9, 54, 55]. This connection motivates the present work, which makes a preliminary step towards understanding the behaviour of singular nonlinear diffusion on manifolds possibly with unbounded curvature; it has to be noticed that in this latter case the weights are locally regular and degenerate only at infinity, see for instance [56].

Since the pioneering paper of Fabes, Kenig and Serapioni [57], weighted (degenerate or singular) elliptic and parabolic equations have been investigated in the linear case $m = 1$, [58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70]; in many cases, the weights are assumed to belong to the "natural" (for second order differential operators) Muckenhoupt class $\mathcal{A}_2$: the reader may notice that the weight $|x|^{-\gamma}$ does not belong to $\mathcal{A}_2$ when $\gamma \in (-\infty, -N)$, a case that we consider here. Our contributions in this direction are quantitative Harnack inequalities and Hölder continuity for weak solutions to linear equations with measurable coefficients. Our results agree with the known results [71, 72, 73, 74] and extend those in some range of parameters.

Ideas of the main results and organization of Part I. The behaviour of solutions to the non-weighted FDE presents strong differences between two ranges: *good fast diffusion range* $m_c < m < 1$ and *very fast diffusion range* $0 < m \leq m_c$: the *critical exponent* being $m_c = (N-2)/N$, see [24, 48]. We show here that also solutions to the WFDE behave quite differently in the two ranges; the *critical exponent* m_c now depends on the weights through γ and β, in the range given by (0.0.11)

$$m_c = \frac{N-(2+\beta)}{N-\gamma} \in (0,1). \tag{0.0.13}$$

Our first main result consists in quantitative upper bounds, see Theorems 0.0.2 and 1.0.1 proven in Chapter 1, which take the form of *local smoothing effects*, that in a simplified form read

$$\sup_{y\in B_R(0)} u\,(t,y) \leq \frac{\overline{\kappa}_1}{t^{(N-\gamma)\vartheta_p}} \left[\int_{B_{2R}(0)} |u_0(y)|^p\ |y|^{-\gamma}\ \mathrm{d}y\right]^{(2+\beta-\gamma)\vartheta_p} + \overline{\kappa}_2 \left[\frac{t}{R^{2+\beta-\gamma}}\right]^{\frac{1}{1-m}},$$

where the exponent $\vartheta_p = [(2+\beta-\gamma)(p-p_c)]^{-1}$ is sharp (see below) and the constants $\overline{\kappa}_1, \overline{\kappa}_2 > 0$, depending only on N, γ and β, have an almost explicit expression. In the so-called *good fast diffusion range*, $m_c < m < 1$, solutions corresponding to $u_0 \in \mathrm{L}^1_{\mathrm{loc}}(|x|^{-\gamma}\,\mathrm{d}x)$, turn out to be locally bounded. In the very fast diffusion range, $0 < m \leq m_c$, a counterexample given in Remark 0.0.3 shows that this is not necessarily true. Indeed, the smoothing effect holds only for data in $\mathrm{L}^p_{\mathrm{loc}}(|x|^{-\gamma}\,\mathrm{d}x)$ with $p > p_c$, the so-called *critical integrability exponent*, defined as

$$p_c = \frac{(1-m)\,(N-\gamma)}{2+\beta-\gamma}. \tag{0.0.14}$$

Note that $\vartheta_p > 0$ whenever $p > p_c$ and that $p_c > 1$ only when $m \in (0, m_c)$. We refer to the monograph [48] for a more detailed exposition of the relevance of such exponents in the non-weighted case $\beta = \gamma = 0$ both for the smoothing estimates and for extinction phenomena.

The second main result is a precise quantitative lower bound for positive solutions, and it shows a remarkable property of WFDE, called “instantaneous positivity”: as it happens in the case without weights, see [75, 24, 25, 76, 77], non zero data immediately produce strictly positive solutions. A simplified version of our result reads:

$$\inf_{x \in B_{2R}(0)} u(t,x) \geq \underline{\kappa} \left[\frac{t}{R^{2+\beta-\gamma}}\right]^{\frac{1}{1-m}} \qquad \text{for any } t \in [0, t_*], \tag{0.0.15}$$

where $t_* = t_*(u_0) \sim \|u_0\|^{1-m}_{\mathrm{L}^1_\gamma(B_R(0))}$, is precisely defined in (0.0.24). We call t_* the *“minimal life time”* of the solution u, following [24], since it represents the amplitude of the time interval in which any nonnegative local solution stays positive. Roughly speaking, if the solution is nonnegative in a small ball, then it becomes instantaneously positive in a bigger ball (expansion of positivity) and for some more time, precisely quantified by the minimal life time t_*; as a consequence, it becomes also Hölder continuous. The above lower bound is somehow optimal, indeed solutions to *fast diffusion*-type equations may extinguish after a finite time $T = T(u_0)$: t_* is a (sharp) lower bound for T; see Chapter 2 for more details.

Our estimates are quantitative and we show an (almost) explicit expression of $\underline{\kappa} > 0$, which depends on the parameters N, m, γ, β, and possibly on u_0 or other geometric quantities. Note that in the good fast diffusion range $m \in (m_c, 1)$, $\underline{\kappa}$ does not depend on the initial data. This does not happen in the linear case $m = 1$, where the lower bound depends on the initial data, and also, it is in contrast with the degenerate case $m > 1$, where the finite speed of propagation forces to wait some time in order to have strict positivity, see [4]. In the very fast diffusion range, $\underline{\kappa} > 0$ also depends on u_0 through $H_p \sim \|u_0\|_{\mathrm{L}^p_\gamma(B_R)} / \|u_0\|_{\mathrm{L}^1_\gamma(B_R)}$, see (0.0.23) for a precise definition.

The proof of (0.0.15) is complex and contains the main new technical novelties of this research. Due to the presence of the weights, the approach developed in [24] for the model equation ($\beta = \gamma = 0$) that relies on moving plane methods (Alexandrov reflection principles) can not be applied. Parabolic De Giorgi-type methods, typically used for equations with coefficients, see [25], can be also applied to the case with weights in appropriate Muckenhoupt classes, see [44, 45]; however, to our best knowledge, these techniques do not provide quantitative results, indeed the constants in the estimates are not always computable. We therefore develop in Chapter 2 a new strategy that allows us to keep the constants explicit.

Upper an lower bounds fairly combine in the form of parabolic Harnack inequalities, our third main result proven in Chapter 3. In the non-weighted case already it has been a longstanding problem to understand which form the Harnack inequality may take (if any) in the very fast diffusion range;

the first answer has been given in [75, 24] and then generalized to other contexts, see the monograph [25]. A simplified version of our Harnack inequalities reads

$$\sup_{x\in B_R(0)} u(t,x) \le \overline{\kappa}_3 \inf_{x\in B_R(0)} u(t,x) \qquad \text{for any} \qquad \frac{t_*}{2} < t < t_*. \tag{0.0.16}$$

where the constant $\overline{\kappa}_3 > 0$ depends on N, γ, β and possibly on the initial data u_0 in some ranges. The minimal life time t_* quantifies the size of the so-called intrinsic cylinders, which roughly speaking represent the right domain where regularity estimates can be stated in a clean form. The size of such intrinsic cylinders depends on the solution itself, due to the singular/degenerate character of the nonlinearity u^m, see [78, 25].

In the very fast diffusion range, $\overline{\kappa}_3$ may depend on u_0 through some weighted L^p norms, and this dependence cannot be avoided in view of explicit counter-examples, see [24, 76] for the non-weighted case. On the other hand, in the good fast diffusion range, $\overline{\kappa}_3$ does not depend on u_0 anymore. In all cases we provide an explicit expression for $\overline{\kappa}_3$, see Remark 0.0.7. It is remarkable that in (0.0.16) we may take the infimum and the supremum at the same time (elliptic-type Harnack inequality), or even at a previous time (backward-type Harnack inequality): this feature is typical of fast diffusion or of singular evolutions [75, 24, 25, 76], and is compatible with the fact that solutions can extinguish in finite time; this happens to be false for general local weak solutions, even for the linear equation $m = 1$, in which case forward Harnack inequalities typically hold. See Theorem 0.0.6 below for a more general statement and remarks.

An important consequence of Harnack-type estimates is Hölder continuity, that we also establish in Chapter 3. A simplified version of our estimates states that there exists $\alpha \in (0,1)$ and $\overline{\kappa}_\alpha > 0$ such that, if $0 \le u \le M_0$ on $(t_0, t_0 + t_*] \times B_{4R_0}(0)$ we have, letting $\sigma = 2 + \beta - \gamma$

$$|u(t,x) - u(\tau,y)| \le \overline{\kappa}_\alpha R_0^{-\sigma} M_0 \big(|x-y|^\sigma + M_0^{m-1}|t-\tau|\big)^{\frac{\alpha}{\sigma}},$$

for all $t_0 + \frac{5}{8}t_* \le t, \tau \le t_0 + \frac{7}{8}t_*$ and all $x, y \in B_{R_0/4}(0)$. See Theorem 0.0.8 for a precise statement. The Hölder exponent α depends on the constant $\overline{\kappa}_3$ of (0.0.16) , and it will be chosen uniformly in the good fast diffusion range, where $\overline{\kappa}_3$ does not depend on u (nor on u_0). On the other hand, in the very fast diffusion case, α may depend on u_0 through some weighted L^p norm: this is somehow natural, since solutions corresponding to data in the weighted L^1 norm may be unbounded, as already discussed above. We provide a (non optimal, but explicit) expression of the exponent α in Chapter 3, and we show that it can vary depending on the cylinder: this may seem strange at a first sight, but indeed it is perfectly reasonable in view of the example given in Remark 0.0.9. We can appreciate here the effect of the weights on the regularity of solutions.

The proof of Hölder continuity in the nonlinear case depends on the regularity results for linear equations with weights. We prove in Chapter 3 both Harnack inequalities and Hölder continuity for weak solutions to linear equations with measurable coefficients, whose prototype is given by $u_t = |x|^\gamma \nabla\cdot\big(|x|^{-\beta} a(t,x)\nabla u\big)$, with $0 < \lambda_0 \le a(t,x) \le \lambda_1$; in our results we keep track of the dependence on λ_0, λ_1 in all constants, as Moser did in the non-weighted case, [74]. We refer to Section 3.1 for more details.

Finally, the Appendix contains the proof of the energy estimates of Chapter 1, the proof of some functional inequalities that we use, together with a number of technical results. We have posponed those long and technical proofs there in order not to break the flow and focus more on the main ideas.

We shall now present the main results and the different scenarios, together with the notation and definition of solutions that we are going to use.

0.0.1 Precise statement of the main results in the different scenarios

In order to state our main results, we need to introduce first some notations and definitions. We will write $a \asymp b$ whenever there exist constants $c_0, c_1 > 0$ such that $c_0 a \le b \le c_1 a$; we let $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

Functional spaces. Let $p \ge 1$ and $\Omega \subseteq \mathbb{R}^N$ be an open connected set with smooth boundary (at least C^2). For any $\gamma \in \mathbb{R}$, μ_γ will denote the measure $\mu_\gamma(\Omega) := \int_\Omega |x|^{-\gamma} dx$ and $\|f\|_{\mathrm{L}^p_\gamma(\Omega)} := \left(\int_\Omega |f|^p |x|^{-\gamma}\,\mathrm{d}x\right)^{\frac{1}{p}}$. We will denote by $\mathrm{L}^p_\gamma(\Omega)$ the weighted L^p-space with respect to μ_γ; it is known that $\mathrm{L}^p_\gamma(\Omega)$ is a Banach space, see [79]. In what follows we will systematically deal with doubly weighted Sobolev spaces, in which the norms of the function and of its derivatives are taken with respect to different measures. In the present weighted setting the usual definition of Sobolev spaces may not yield to a complete space, see [79], therefore, we will follow the ideas of Fabes, Kenig and Serapioni [57], see also [80], and to avoid technical difficulties, we shall always assume that the parameters γ and β satisfy assumption (0.0.11). We define $H^1_{\gamma,\beta}(\Omega)$ to be the closure of $C^\infty\left(\bar{\Omega}\right)$ with the topology given by the norm $\|\phi\|^2_{H^1_{\gamma,\beta}(\Omega)} = \|\phi\|^2_{2,\gamma} + \|\nabla\phi\|^2_{2,\beta}$, and $\mathcal{D}_{\gamma,\beta}(\Omega)$ to be the closure of $C^\infty_c(\Omega)$ under the norm $\|\phi\|_{\mathcal{D}_{\gamma,\beta}(\Omega)} = \|\nabla\phi\|_{2,\beta}$. This procedure leads to the definition of a complete space in which functions have a unique weak gradient, obtained by approximation. Without any further assumption on the weights, the limit of such approximation may fall out of $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$, see [57, section 2 and 3] and [80]. As a consequence, solutions to WFDE need to be considered in a suitable weak sense, as follows.

Definition 0.0.1 (Weak and strong solutions)**.** *Let $Q = (T_0, T] \times \Omega \subseteq (0,\infty) \times \mathbb{R}^N$. A function $u : Q \to \mathbb{R}$ is a local weak solution to equation (WFDE) in Q if*

$$u \in C_{\mathrm{loc}}((T_0,T); \mathrm{L}^2_{\gamma,\mathrm{loc}}(\Omega)) \qquad \text{and} \qquad u^m \in \mathrm{L}^2_{\mathrm{loc}}((T_0,T); H^1_{\gamma,\beta,\mathrm{loc}}(\Omega)), \tag{0.0.17}$$

and the following identity holds,

$$\begin{aligned}\int_\Omega & \left[u(t_2,x)\phi(t_2,x) - u(t_1,x)\phi(t_1,x)\right]|x|^{-\gamma}\,\mathrm{d}x \\ &= \int_{t_1}^{t_2}\int_\Omega u\phi_t\ |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_\Omega \nabla u^m \cdot \nabla\phi\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{0.0.18}$$

for every open subset $[t_1,t_2] \times K \subset Q$ and for any test function ϕ such that

$$\phi \in W^{1,2}_{\mathrm{loc}}((T_0,T); \mathrm{L}^2_\gamma(K)) \cap \mathrm{L}^2_{\mathrm{loc}}((T_0,T); \mathcal{D}_{\gamma,\beta}(K)).$$

A local strong solution to WFDE is a local weak solution with $u_t \in \mathrm{L}^1_{\mathrm{loc}}((0,T)\,; \mathrm{L}^1_{\gamma,\mathrm{loc}}(\Omega))$.
A local weak (or strong) sub (resp. super) solution satisfies (0.0.18) *with $\le$ (resp. $\ge$) for any nonnegative test function in the same class.*

About the class of solutions. Most of our results will be proven for local strong solutions: lengthy (but nowadays standard) approximation procedures allow one to extend our results to a wider class of solutions, the so-called limit solutions [4], sometimes also called SOLA, Solutions Obtained by Limit of Approximations [81, 82]. In particular, our results apply to weak solutions in the sense of the above definition. Such approximations are often long and technical in the framework of local solutions, but easier when dealing with global problems, like Cauchy, Dirichlet, Neumann, Robin, or even for solutions of large problems (Dirichlet problems whose solutions go to $+\infty$ on the lateral boundary). We shall say that most of the weaker concepts of solutions are included in the so-called class of limit solutions, i.e. limit of strong solutions, for which our estimates apply by a simple limiting process.

Weights and different scenarios. Our results concern quantitative a priori upper estimates of local type. We will consider a fixed cylinder of reference $Q = (T_0, T] \times \Omega \subseteq (0, \infty) \times \mathbb{R}^N$, as in Definition 0.0.1, and the estimates will take place on a smaller cylinder, typically sufficiently far from the boundary of Ω. Due to the lack of translation invariance of the weights, we need to find the right quantity that takes into account for change of geometry. Following [65, 83, 44, 45] we define for $x_0 \in \mathbb{R}^N$ and $R > 0$:

$$\rho_{x_0}^{\gamma,\beta}(R) := \left(\int_{B_R(x_0)} |x|^{(\beta-\gamma)\frac{N}{2}} \, \mathrm{d}x \right)^{\frac{2}{N}} . \tag{0.0.19}$$

The inequality

$$\overline{\kappa}_{16}^{-1} \, \rho_{x_0}^{\gamma,\beta}(R) \le R^2 \frac{\mu_\gamma(B_R(x_0))}{\mu_\beta(B_R(x_0))} \le \overline{\kappa}_{16} \, \rho_{x_0}^{\gamma,\beta}(R) \,,$$

is proven in Lemma 3.4.1. Roughly speaking, the scaling properties of the equation change from $R^{2+\beta-\gamma}$ to (a multiple of) R^2, when we are considering problems around the origin or far from it, respectively. There are at least four possible scenarios, see figure (1) on page 27:

(a) When $x_0 = 0$ and $R_0 > 0$, we have $\rho_{x_0}^{\gamma,\beta}(R_0) \sim R_0^{2+\beta-\gamma}$.

(b) When $x_0 \neq 0$ and $0 \in B_{R_0}(x_0)$, we have $\rho_{x_0}^{\gamma,\beta}(R_0) \sim R_0^{2+\beta-\gamma}$.

(c) When $x_0 \neq 0$, $0 \notin B_{R_0}(x_0)$ and $R_0 > |x_0|/2$, namely x_0 is relatively far from the origin but the singularity is still felt by the equation, and in this case we have $\rho_{x_0}^{\gamma,\beta}(R_0) \sim R_0^{2+\beta-\gamma}$.

(d) When $x_0 \neq 0$, $0 \notin B_{R_0}(x_0)$ and $0 < R_0 \le |x_0|/2$. This is the case where x_0 is relatively far from the origin and does not heavily affect the geometry of the parabolic cylinders. In this case we are essentially dealing with a nonlinear singular parabolic equation (governed by the nonlinearity u^m) and where the diffusion is driven by a uniformly elliptic operator; more specifically the standard (non-intrinsic) parabolic cylinders, depend on the ellipticity constants which in turn are proportional to $|x_0|^{\beta-\gamma}$, more precisely $\rho_{x_0}^{\gamma,\beta}(R_0) \sim R_0^2 \, |x_0|^{\beta-\gamma}$; note that all these latter quantities are bounded and bounded away from zero in this case; see for instance [74] for the linear case and [24, 25] for the nonlinear case.

We will focus only on the cases (a), (b) and (c) in which we have novel results, and where the geometry of the weights really plays a role; as already mentioned, the case (d) follows from nowadays standard results. For the sake of simplicity, from now on, we will always make one of the following assumptions on the ball $B_{R_0}(x_0)$ where our local estimates will take place:

(1) Let $x_0 = 0$ and any $R_0 > 0$, or

(2) Let $x_0 \neq 0$ and $|x_0|/32 \le R_0 \le |x_0|/16$, or

(3) Let $x_0 \neq 0$ and $(5/2)|x_0| \le R_0 \le 4|x_0|$.

We notice that under the assumptions (1), (2) or (3) a simple calculation shows that (see proof of Lemma 3.4.1)

$$\overline{\kappa}_{17}^{-1} R^{2+\beta-\gamma} \le R^2 \frac{\mu_\gamma(B_R(x_0))}{\mu_\beta(B_R(x_0))} \le \overline{\kappa}_{17} R^{2+\beta-\gamma} \,, \tag{0.0.20}$$

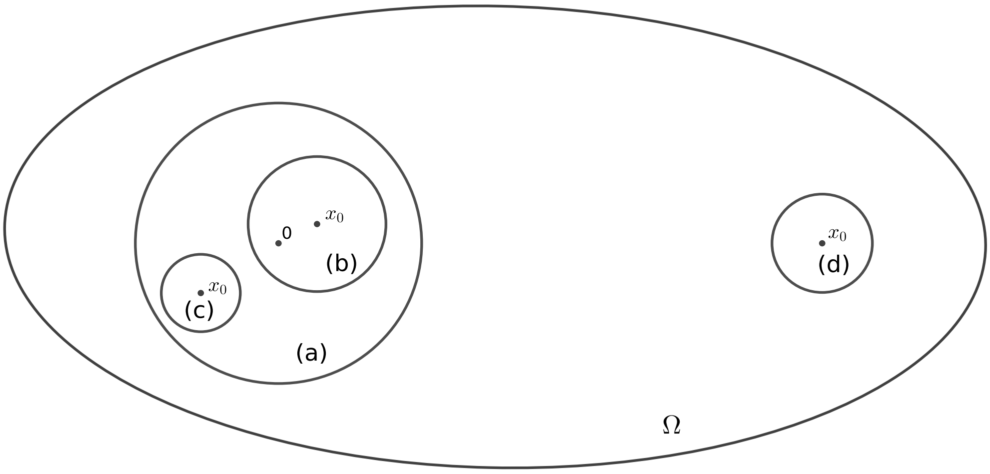

Figure 1: A representation of possible scenarios (a), (b), (c) and (d).

where the constant $\overline{\kappa}_{17} > 0$ depends only on N, γ, β. Before stating our main results we first need to introduce the following parameters whose role has been already explained above (recall that p_c is defined in (0.0.14))

$$\sigma = 2 + \beta - \gamma \qquad \text{and} \qquad \vartheta_p = \frac{1}{\sigma p - (N-\gamma)(1-m)} = \frac{1}{\sigma(p-p_c)}. \tag{0.0.21}$$

Theorem 0.0.2 (Local Upper Bounds). *Let u be a nonnegative local strong (sub)solution to WFDE on the cylinder $\Omega \times (0,T]$. Let moreover $p \geq 1$ if $m \in (m_c, 1)$ and $p > p_c$ if $m \in (0, m_c]$. Let $B_{2R_0}(x_0) \subset \Omega$ and assume that $B_{R_0}(x_0)$ satisfies either (1),(2) or (3). Then there exist $\overline{\kappa}_1, \overline{\kappa}_2 > 0$ such that for any $t \in (0,T]$ we have*

$$\sup_{y \in B_{R_0}(x_0)} u(t,y) \leq \frac{\overline{\kappa}_1}{t^{(N-\gamma)\vartheta_p}} \left[\int_{B_{2R_0}(x_0)} |u_0(y)|^p \frac{\mathrm{d}y}{|y|^\gamma}\right]^{\sigma\vartheta_p} + \overline{\kappa}_2 \left[\frac{\mu_\beta(B_{R_0}(x_0))}{\mu_\gamma(B_{R_0}(x_0))} \frac{t}{R_0^2}\right]^{\frac{1}{1-m}}, \tag{0.0.22}$$

where ϑ_p and σ are defined as in (0.0.21). *The constants $\overline{\kappa}_1, \overline{\kappa}_2$ depend only on N, γ and β.*

Chapter 1 contains the proof of Theorem 1.0.1, which implies the above theorem as a particular case.

Remark 0.0.3. (i) The above smoothing effect may fail when $m < m_c$, if we choose exponents $p < p_c$. Indeed, there is an explicit counterexample to the above $\mathrm{L}^p_\gamma \to \mathrm{L}^\infty$ smoothing effect for solutions with initial data in $\mathrm{L}^p_{\gamma,\mathrm{loc}}$ with $p < p_c$, given by the following function:

$$U(t,x) = c\,(T-t)^{\frac{1}{1-m}} |x|^{-\frac{\sigma}{1-m}},$$

where $c = c(m, N, \gamma, \beta)$ is chosen in such a way that U becomes a local solution to WFDE in the cylinder $(0,T) \times \mathbb{R}^N$. In the non-weighted case $\beta = \gamma = 0$, the above counterexample was shown in [24, 48].

(ii) It is worth noticing that when $\gamma = 2+\beta$ or when $\sigma = 0$, i.e. outside our range of parameters (0.0.11), also in the linear case the smoothing effect fails: this has been proved in [65] by means of counterexamples.

(iii) The above upper bound (0.0.22) contains two terms, the first takes into account the influence of the initial data, while the second takes into account the "worst case scenario", that happens when the local weak solution comes from the so-called large solutions, namely solutions to the Dirichlet problem which go to $+\infty$ at the lateral boundary, see for instance [75, 24].

Our second result concerns quantitative positivity estimates. In order to state our main positivity results, we need to introduce first the following intrinsic quantities, for $p \geq 1$:

$$\begin{aligned} H_p\left(f, x_0, R\right) &:= \frac{\mu_\gamma(B_R(x_0))^{\sigma\vartheta_p}}{\mu_\gamma(B_R(0))^{\sigma\vartheta_p}} \left[\frac{\mu_\gamma(B_R(x_0)) \left(\int_{B_R(x_0)} f^p |x|^{-\gamma}\,\mathrm{d}x\right)^{\frac{1}{p}}}{\mu_\gamma(B_R(x_0))^{\frac{1}{p}} \int_{B_R(x_0)} f |x|^{-\gamma}\,\mathrm{d}x}\right]^{p\sigma\vartheta_p}, \qquad (0.0.23)\\ \widetilde{H}_p := \widetilde{H}_p(f, x_0, R) &:= 1 + \left(\frac{|x_0|}{R} \vee 1\right)^{\beta-\gamma} H_p(f, x_0, R)^{1-m} \geq 1\,, \end{aligned}$$

where ϑ_p and σ are defined as in (0.0.21); we notice that in the cases (1), (2) and (3) the quantities $|x_0|/R_0$ and $\mu_\gamma(B_R(x_0))^{\sigma\vartheta_p}\,\mu_\gamma(B_R(0))^{-\sigma\vartheta_p}$, become independent of x_0, R. The above quantity is an adaptation to the weighted case of a similar one introduced in [24] and it plays an essential role in the positivity estimates: in particular, $\widetilde{H}_p$ encodes the geometric information of the weights which is relevant in the estimates. An important aspect of these quantities, that will play an important role in our main results, is that both H_p and $\widetilde{H}_p$ are scaling invariant, with respect to the natural scaling of the equation, see for instance formula (2.9.2). Finally, we would like to emphasize that in the good fast diffusion range, i.e. when we can choose $p = 1$, H_p (hence $\widetilde{H}_p$) does not depend on f:

$$H_1(f, x_0, R) = \frac{\mu_\gamma(B_R(x_0))^{\sigma\vartheta_1}}{\mu_\gamma(B_R(0))^{\sigma\vartheta_1}}.$$

We are now in the position to state our main positivity result.

Theorem 0.0.4 (Local Lower Bounds). *Let u be a nonnegative local strong (super)solution to WFDE on $(0,T)\times\Omega$ and let $0 \leq u_0 \in \mathrm{L}^p_{\gamma,\mathrm{loc}}(\Omega)$ with $p \geq 1$ if $m \in (m_c, 1)$ and $p > p_c$ if $m \in (0, m_c]$. Let $B_{4R}(x_0) \subseteq \Omega$ and assume that $B_R(x_0)$ satisfies either (1), (2) or (3). Define the minimal life time t_* as*

$$t_* = t_*(u_0, x_0, R) = \kappa_*\, R^\sigma \frac{\|u_0\|^{1-m}_{\mathrm{L}^1_\gamma(B_R(x_0))}}{\mu_\gamma(B_R(x_0))^{1-m}}\,. \qquad (0.0.24)$$

Then, there exists $\underline{\kappa} = \underline{\kappa}\big(H_p(u_0, x_0, R), R, N, m, \gamma, \beta\big) > 0$ such that

$$\inf_{x\in B_{2R}(x_0)} u(t,x) \geq \underline{\kappa}\left[\frac{\mu_\beta(B_R(x_0))}{\mu_\gamma(B_R(x_0))}\,\frac{t}{R^2}\right]^{\frac{1}{1-m}} \qquad \textit{for any } t \in [0, t_*\wedge T]. \qquad (0.0.25)$$

The constant $\kappa_* > 0$ depends on N, m, γ, β and it is given in Corollary 2.8.2; $\underline{\kappa}$ has an (almost) explicit expression given in (2.8.10) *and depends on $H_p(u_0, x_0, R)$ only when $m \in (0, m_c]$.*

Remark 0.0.5. (i) Roughly speaking, the above lower bound (0.0.25) shows that any bounded nonnegative solution becomes instantaneously (strictly) positive on a whole time interval $(0, t_*(u_0, x_0, R)]$. This result will allow us to give *an estimate on the size of the intrinsic cylinders*, which are the natural domain for Harnack and Hölder continuity estimates, see also Chapter 3. We construct intrinsic cylinders inside arbitrary ones: in the literature this is often an assumption, cf. [25].

(ii) All the constants are computable: from the expression (2.8.10) of $\underline{\kappa}$, we deduce that when $\widetilde{H}_p$, defined in (0.0.23), is large enough, then there is a constant $c_1 > 0$ depending on N, m, p, β, γ such that

$$\underline{\kappa} \asymp \widetilde{H}_p^{-\frac{c_1 \widetilde{H}_p^{1/2}}{m(1-m)}},$$

(iii) As already mentioned above, when $m_c < m < 1$, the constant H_p does not depend on u_0 anymore, hence formula (0.0.25) provides an absolute lower bound, i.e. independent of u and u_0 :

$$\inf_{x \in B_{2R}(x_0)} u(t,x) \geq \underline{\kappa}' \left[\frac{\mu_\beta(B_R(x_0))}{\mu_\gamma(B_R(x_0))} \frac{t}{R^2} \right]^{\frac{1}{1-m}} \qquad \text{for any } t \in [0, t_* \wedge T],$$

where $\underline{\kappa}'$ only depends on N, m, γ, β. However the presence of u_0 is still felt through $t_* \sim \|u_0\|_{\mathrm{L}^1_\gamma}^{1-m}$.

(iv) Chapter 2 contains a detailed proof of Theorem 0.0.4, which is the major technical contribution of this work. Our proof applies also to the non-weighted case $\gamma = \beta = 0$, and we recover the result of [24] with a different proof.

Our quantitative upper and lower bounds fairly combine into Harnack-type inequalities.

Theorem 0.0.6 (Harnack Inequalities). *Let u be a nonnegative local strong solution to WFDE on $(0,T) \times \Omega$ and let $0 \leq u_0 \in \mathrm{L}^p_{\gamma,\mathrm{loc}}(\Omega)$ with $p \geq 1$ if $m \in (m_c, 1)$ and $p > p_c$ if $m \in (0, m_c]$. Let $B_{8R}(x_0) \subseteq \Omega$, $t_0 \in [0,T)$ and assume that $B_{2R}(x_0)$ satisfies either (1), (2) or (3). Define*

$$t_* = t_*(u(t_0), x_0, 2R) = \kappa_*(2R)^\sigma \mu_\gamma(B_{2R}(x_0))^{m-1} \|u(t_0)\|_{\mathrm{L}^1_{B_{2R}(x_0)}}^{1-m}.$$

Then, for any $\varepsilon \in (0,1)$, there exists $\overline{\kappa}_3 > 0$ such that for any $t, t \pm \theta \in [t_0 + \varepsilon t_*(t_0), t_0 + t_*(t_0)] \cap (0,T)$

$$\sup_{x \in B_R(x_0)} u(t,x) \leq \overline{\kappa}_3 \inf_{x \in B_R(x_0)} u(t \pm \theta, x). \tag{0.0.26}$$

The constants $\kappa_*, \overline{\kappa}_3 > 0$ always depend on N, m, γ, β and are given in (2.8.5) *and* (3.2.2) *respectively; $\overline{\kappa}_3$ may also depend on R, x_0 and ε, and, when $0 < m \leq m_c$, it depends on $H_p(u_0, x_0, 2R)$ defined in* (0.0.23)*.*

Remark 0.0.7. (i) In (0.0.26) we may take the infimum and the supremum at the same time (elliptic-type Harnack inequality, $\theta = 0$); we can even take the infimum at a previous time (backward-type Harnack inequality, $\theta < 0$): both inequalities are in contrast with the classical parabolic Harnack inequality valid for the linear heat equation ($m = 1$), which needs to be forward in time (infimum at a later time, $\theta > 0$), [84, 73, 85]. Indeed, elliptic and backward Harnack inequalities are typical features of fast diffusion equations, as already observed in [24, 25]. They are false in general for the Heat Equation ($m = 1$) and for the Porous Medium Equation ($m > 1$), when dealing with local solutions (i.e. regardless of the boundary conditions), or in the case of solutions to the Cauchy problem posed on $\mathbb{R}^N$. However for solutions to the homogeneous Dirichlet problem, also when $m \geq 1$, elliptic and backward inequalities have been proven, see [86, 87, 88, 89, 90].

(ii) Our result is quantitative, in the sense that all the constants are computable: from the expression (3.2.2) of $\overline{\kappa}_3$, when $\widetilde{H}_p$ is large enough, we deduce

$$\overline{\kappa}_3 \asymp \varepsilon^{-\frac{\sigma p \vartheta_p}{1-m}} H_p(u_0, x_0, 2R)\, \widetilde{H}_p^{\frac{c\, \widetilde{H}_p^{1/2}}{m(1-m)}},$$

where $\widetilde{H}_p$ is given in (0.0.23), and $c > 0$ only depends on N, m, p, β, γ.

(iii) Recall that in the good fast diffusion range, we can choose $p=1$ and obtain a more classical form of the above Harnack inequality i.e. with the constant independent of u_0:

$$\sup_{x\in B_R(x_0)} u(t,x) \le \overline{\kappa}_3' \inf_{x\in B_R(x_0)} u(t\pm\theta,x),$$

with t_* as in the formula above (0.0.26) and where $\overline{\kappa}_3'$ depends only on N, m, p, γ, β and possibly R, x_0, ε. In the very fast diffusion range we can not eliminate the dependence on u_0 in the above Harnack inequalities (0.0.26); indeed, explicit counterexamples as in the non-weighted case can be constructed, see [24, 76].

Our last main result concerns Hölder continuity estimates, and the proof relies on some results for linear weighted equations that we describe later

Theorem 0.0.8 (Interior Hölder Continuity). *Let u be a nonnegative local weak solution to the* (WFDE) *on $Q := [0,T)\times\Omega$. Let $t_0\in[0,T)$ and $B_{16R_0}(x_0)\subset\Omega$ and assume that $B_{4R_0}(x_0)$ satisfies either (1), (2) or (3). If $u\le M_0<\infty$ on $(t_0, T\wedge(t_0+t_*)]\times B_{4R_0}(x_0)$ with $t_*=t_*(u(t_0),x_0,4R_0)$ as in* (0.0.24)*, letting*

$$D_0 := 1\wedge\overline{\kappa}_{19}^{-2}\,(T\wedge t_*/8)^{1/\sigma}\wedge\overline{\kappa}_{19}^{-2}\left(\rho_{x_0}^{\gamma,\beta}\right)^{-1}(T\wedge t_*/8)\,,$$

with a suitably small $\overline{\kappa}_{19}>0$ as in (3.4.2)*, then there exist $\alpha\in(0,1)$ and $\overline{\kappa}_\alpha'>0$, such that*

$$|u(t,x)-u(\tau,y)| \le \frac{\overline{\kappa}_\alpha'\,M_0}{D_0^\alpha\left(R_0\wedge R_0^{\frac{\sigma}{2\vee\sigma}}\right)^\alpha}\left(|x-y|+M_0^{\frac{m-1}{2\vee\sigma}}|t-\tau|^{\frac{1}{2\vee\sigma}}\right)^\alpha, \tag{0.0.27}$$

for all $t,\tau\in\left[t_0+\frac{5}{8}t_*, t_0+\frac{7}{8}t_*\right]\cap(0,T)$ and for all $x,y\in B_{R_0}(x_0)$. The constants $\alpha\in(0,1)$ and $\overline{\kappa}_\alpha'>0$ depend on $N, m, \gamma, \beta, H_p(u(t_0), x_0, 4R_0)$ and possibly on R_0, x_0.

Remark 0.0.9. (i) We provide a (non optimal, but explicit) expression of the exponent α in Chapter 3, and we show that it can vary depending on the base point x_0 and on the radius R_0: this may seem strange at a first sight, but indeed it is perfectly reasonable in view of the following example. Consider the Cauchy problem on the whole space: the fundamental (or Barenblatt) solution, has a selfsimilar form $B(t,x)=t^aF(|x|t^{-b})$ where $F(|x|)=A(D+|x|^\sigma)^{1/(m-1)}$, see [18, 19]. Clearly this explicit solution is merely Hölder continuous at zero when $\sigma\in(0,1]$, is $C^{1,\alpha}$ when $\sigma\in(1,2]$ and so on, but such a solution is always $C^\infty(\mathbb{R}^N\setminus\{0\})$. We can appreciate here the effect of the weights on the regularity of solutions. Again, it is worth noticing that when $\sigma=0$, Hölder continuity fails, as well as the upper bounds, see Remark 0.0.3 (ii).

(ii) We have decided to state the Theorem in this simplified form, to focus on the main result. Indeed it is quite easy to show that it holds for $t,\tau\in\left[t_0+\frac{5}{7}\varepsilon t_*, t_0+\varepsilon t_*\right]\cap(0,T)$, for any $\varepsilon\in(0,7/8)$, paying the price of having a dependence on ε in the constant $\overline{\kappa}_\alpha$, as it happens for the Harnack inequality of Theorem 0.0.6.

(iii) The above theorem is stated in a general form emphasizing the fact that bounded solutions are C^α on a smaller intrinsic cylinder, whose size depends both on t_* (i.e. on the L^1_γ norm) and on the L^∞ bound M_0. A closer inspection of the proof reveals that by slightly changing t_* to $t_*=t_*(u(t_0),x_0,8R_0)$, and using the upper bounds of Theorem 0.0.2 or 1.0.1, we can choose $M_0=cH_p(u(t_0),x_0,8R_0)(t_*/R_0^\sigma)^{1/1-m}$, by means of the same computation (3.2.1) as in the proof of Theorem 0.0.6.

Also, the exponent α depends on H_p and t_* in a quantitative way,

$$\alpha \sim \exp\left(-\frac{c_6}{t_*} H_p^{\frac{c_7(1-m)}{m} H_p^{(1-m)/2}}\right)$$

where $c_i > 0$ only depend on N, m, p, β, γ. See the end of the proof of Theorem 0.0.8, for more details. Therefore, the size of the intrinsic cylinders in the good fast diffusion range can be chosen to depend only on t_* (i.e. on the $\mathrm{L}^1_{\gamma,\mathrm{loc}}$ norm of $u(t_0)$), while in the very fast diffusion range it has to depend also on H_p. The same happens for the exponent α. This reveals a typical feature of the fast diffusion equation, for which there are strong differences between the two regimes. We last notice that such quantities are stable in the limit $m \to 1^-$, in which case we recover the linear results. On the other hand, by the above formulae, it is also clear that $\alpha \to 0^+$ when either $m \to 0^+$ or $H_p \to +\infty$. This is compatible with the fact that solutions to the Dirichlet problem with $m = 0$ (i.e. the logarithmic diffusion) extinguish immediately, cf. [91, 48], and with the fact that if $H_p = +\infty$ the solution maybe unbounded.

(iv) We have already mentioned above that the optimal Hölder exponent α is not known, since α in general has to depend on R_0, x_0 and u_0. However, in some particular cases, α can be chosen uniformly in the whole range $m \in (0, 1)$. This happens for instance in the Cauchy problem on the whole space, when we deal with the class of solutions trapped between two Barenblatt, in which case H_p can be shown to be a suitable constant, see for instance [18, 19]. Note that in the latter case, solutions are C^α around the origin, and classical elsewhere.

(v) *About a uniform Hölder exponent.* A closer inspection of the proof reveals that indeed it is possible to choose a uniform Hölder exponent: in the good fast diffusion range, we can let $p = 1$ and choose $M_0 \asymp \|u(t_0)\|_{\mathrm{L}^1_\gamma(B_{4R_0})}$ to obtain an (explicit) exponent α which depends only on N, γ, β.

Harnack inequalities and Hölder continuity in the linear case. The study of quantitative regularity estimates for linear parabolic equations with measurable coefficients began with Moser [74]. We show in Section 3.1 analogous quantitative Harnack inequalities and Hölder continuity for weak solutions to linear equations with degenerate/unbounded coefficients, whose prototype is given by $u_t = |x|^\gamma \nabla \cdot \left(|x|^{-\beta} a(t,x) \nabla u\right)$, with $0 < \lambda_0 \le a(t,x) \le \lambda_1$; in our estimates, we keep track of the dependence on λ_0, λ_1. We do not claim originality for these results, indeed in many ranges of parameters they were already known [62, 63, 64, 65, 72, 71]; however we did not find in the literature the quantitative result that we needed, hence we sketch the proof in Section 3.1. The motivation for this analysis comes from the application to nonlinear equations: our proof of the Hölder continuity for solutions to WFDE heavily depends on the linear estimates.

0.0.2 Possible generalizations.

Other ranges of parameters. Formally our results extend to a wider zone of parameters, namely zone (II) in Figure 2 on page 33, which amounts to require: $\gamma > N$ and $\gamma - 2 > \beta \ge \frac{N-2}{N}\gamma$, note that in this case the weight $|x|^{-\gamma}$ is not integrable at $x = 0$, also $|x|^{-\beta}$ is allowed to be not integrable and $\sigma = 2 + \beta - \gamma < 0$. Allowing this range of parameters would require more technical results about the weighted functional spaces and inequalities involved in our proofs: we have decided to not treat this case here, since a rigorous proof would require a significant amount of technical results that would increase the length of this work. For the sake of simplicity we assume that $N \ge 3$, but our method works with straightforward modifications also when $N = 1, 2$.

More general equations. We can allow for more general weights, equations and nonlinearities. For instance, all the results of Part I easily generalize to nonnegative solutions of

$$w_\gamma^{-1}(x)u_t = \sum_{i,j=1}^{N} \partial_i \left(A_{i,j}(x)\partial_j u^m + B_i(x)u^m\right),$$

with

$$w_\gamma(x) \asymp |x|^\gamma\,, \quad 0<\lambda_0|x|^{-\beta}|\xi|^2 \le \sum_{i,j=1}^{N} A_{i,j}(x)\xi_i\xi_j \le \lambda_1|x|^{-\beta}|\xi|^2\,, \text{ and } \quad |B_i(x)| \le \lambda_1 |x|^{-\frac{\beta+\gamma}{2}}$$

for some constants $0<\lambda_0\le\lambda_1$; we can also translate the singularity at another point $x_0\neq 0$. Note that the upper bounds extend also to signed solutions, recalling that in this case we have to work with odd nonlinearities $u^m := |u|^{m-1}u$.

A close inspection of the proofs reveals that all our results can be adapted, with some extra work, to nonlinearities $F(u)$ with $F\in C^1(\mathbb{R}\setminus\{0\})$ with $F/F'\in \mathrm{Lip}(\mathbb{R})$ such that there exist $0<m_0\le m_1$ such that

$$\frac{1}{m_1} \le \left(\frac{F}{F'}\right)' \le \frac{1}{m_0}.$$

It is often convenient to take $m_0, m_1\in(0,1)$, but we can also allow $m_1\ge 1$; the above assumption guarantees that $t\mapsto t^{\frac{m_0}{m_1(1-m_0)}}u(t,\cdot)$ is monotone non-increasing, see for instance [92]; as a consequence, all the proofs of Part I can be repeated with minor modifications. The rough idea is that our results extend to nonlinearities that behave like a concave power, $F(u)\asymp |u|^{m_0-1}u$ for $|u|\sim 0$, and behave like another (not necessarily concave) power $F(u)\asymp |u|^{m_1-1}u$ for $|u|\gg 1$. We stress that on one hand the qualitative results are still true (boundedness, positivity and continuity) also for more general nonlinearities. On the other hand, although qualitatively the same, the quantitative results shall have a quite different form, namely the exponents in the estimates and the dependence on the data in the constants and in the estimates may change in function of m_0, m_1; one advantage of the present method is that all the quantities can be controlled in a quantitative way.

The space-time estimates of Theorem 1.3.1 and Proposition 2.4.1 in its space-time form (2.4.11), can be extended to even more general nonlinear operators of the form

$$w_\gamma^{-1}(x)u_t = \nabla\cdot A(t,x,u,\nabla u) + B(t,x,\nabla u,u),$$

with w_γ as above and

$$|A(t,x,u,\nabla u)| \le \lambda_1|x|^{-\beta}\left|\nabla|u|^m\right| \quad \text{and} \quad B(t,x,u,\nabla u)\cdot\nabla u \ge \lambda_0|x|^{-\beta}\left|\nabla|u|^m\right|^2.$$

Again, the assmptions on the power-like nonlinearity can be weakened, as above, and we can allow a concave F with $F(u)\asymp|u|^{m-1}u$ for $u\sim 0$, and regular outside zero. As for the lower order term B, the typical assumption would be $|B(t,x,u,\nabla u)| \le \lambda_1|x|^{-\beta}\left|\nabla|u|^m\right| + \lambda_1^2 w_\gamma^{-1}(x)|u|^m$, but it can be weakened. On one hand, it is possible to obtain upper bounds in a refined form like Theorem 0.0.2 or 1.0.1 also in this generality. On the other hand, precise lower bounds like in Theorem 0.0.4 or 2.0.1, are not easily extended in this degree of generality: the major technical difficulty is represented by the absence of time monotonicity for solutions to a homogeneous Dirichlet problems, namely that $t\mapsto t^{\frac{1}{1-m}}u(t,\cdot)$ is monotone non-increasing.

Finally, our methods can be adapted to hold also on Riemannian manifolds; we can possibly allow for manifolds with unbounded curvature, as already mentioned, and this partially motivates the present research.

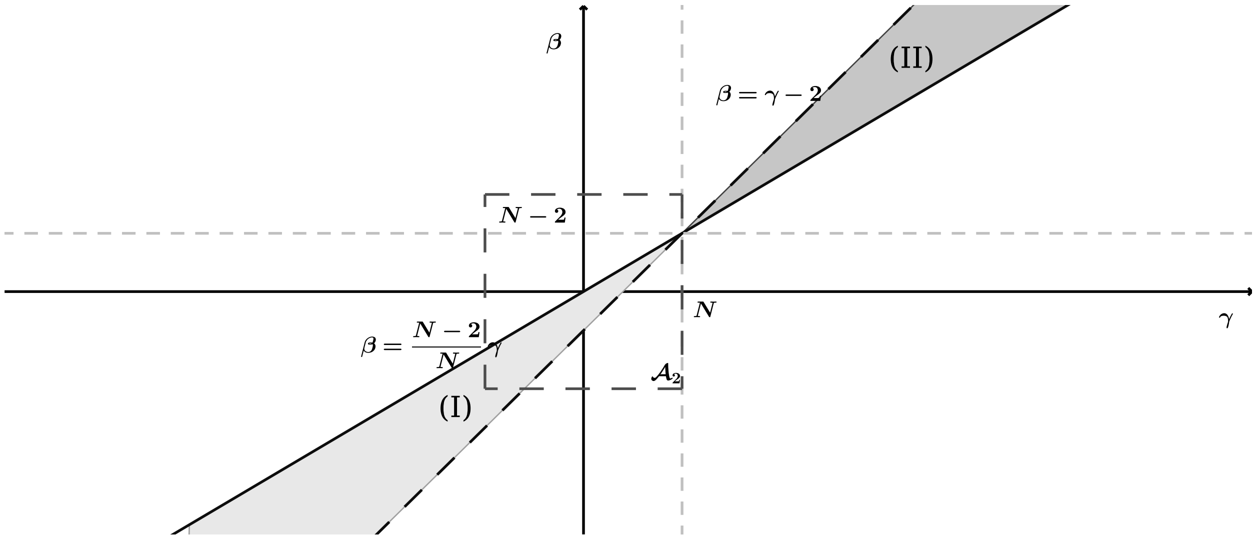

Figure 2: In light grey the region (I) of parameters β and γ as in (0.0.11), where all of our results hold. Note that a big part of the region falls outside the $\mathcal{A}_2$ region. The region (II) in dark grey correspond to a range of parameters that we do not treat, but where our results formally apply, as discussed in Subsection 0.0.2

0.0.3 Weighted Functional Inequalities

In order to study regularity properties of the solution to WFDE a key point in our approach is represented by weighted functional inequalities, that we briefly recall here. For any $\gamma < N$, consider the measure $\mu_\gamma(B) := \int_B |x|^{-\gamma}\,\mathrm{d}x$, which is known to be doubling, i.e.

$$\mu_\gamma(B_{2R}) \leq D_\gamma \mu_\gamma(B_R), \tag{0.0.28}$$

where B_R is a ball contained in $\mathbb{R}^N$ and the constant D_γ depends only on the dimension N and the parameter γ, see [80, Chp. 15]. On the whole $\mathbb{R}^N$, there is a celebrated family of interpolation inequalities, the so-called Caffarelli-Kohn-Nirenberg inequalities [5], that we state hereafter in a special case, namely as in (0.0.12). Let γ, β as in (0.0.11), then there exists a constant $\overline{S}_{\gamma,\beta} > 0$ such that for any $f \in C_c^\infty\left(\mathbb{R}^N\right)$ the following inequality holds

$$\|f\|_{\mathrm{L}^{r^*}_\gamma(\mathbb{R}^N)} \leq \overline{S}_{\gamma,\beta}\|\nabla f\|_{\mathrm{L}^2_\beta(\mathbb{R}^N)} \qquad \text{where} \qquad r^* = 2\frac{N-\gamma}{N-(2+\beta)}\,, \tag{CKNI}$$

where the weighted L^p norms are defined in subsection 0.0.1. This family of inequalities contains both the classical Sobolev inequality ($\gamma = \beta = 0$) and the Hardy inequality ($\beta = \gamma - 2$), cf. [93]. In our range of parameters (0.0.11) (see Figure 2) we always have $r^* \in [2, 2N/(N-2)]$.

Proposition 0.0.10. *Let $N \geq 3$, γ, β be as in* (0.0.11)*, r^* be as in* (CKNI) *and σ as in* (0.0.21)*. Let $x_0 \in \mathbb{R}^N$ and $R > 0$. Then, there exists a constant $S_{\gamma,\beta} > 0$ depending only on N, β, γ, such that for any $f \in H^1_{\gamma,\beta}(B_R(x_0))$ the following inequality holds*

$$\|f\|_{\mathrm{L}^{r^*}_\gamma(B_R(x_0))} \leq S_{\gamma,\beta}\left(\|\nabla f\|_{\mathrm{L}^2_\beta(B_R(x_0))} + \mu_\gamma(B_R(x_0))^{\frac{-\sigma}{2(N-\gamma)}}\|f\|_{\mathrm{L}^2_\gamma(B_R(x_0))}\right)\,. \tag{CKNI2}$$

The above (CKNI2) will play the role of the classical Sobolev inequality in the proof of both upper and lower bounds, hence we have given a short proof in Appendix-B.

Another essential tool in our proofs will be the following weighted Poincaré inequality.

Proposition 0.0.11 (Poincaré Inequality). *Let be $N \geq 3$ and $\gamma, \beta < N$ as in* (0.0.11), $x_0 \in \mathbb{R}^N$ *and $R > 0$. Then there exists a constant $P_{\gamma,\beta} > 0$ such that for any $\phi \in H^1_{\gamma,\beta}(B_R(x_0))$ we have*

$$\left(\frac{1}{\mu_\gamma(B_R(x_0))}\int_{B_R(x_0)} |\phi - \overline{\phi_\gamma}|^2 \frac{\mathrm{d}x}{|x|^\gamma}\right)^{\frac{1}{2}} \leq P_{\gamma,\beta}\; R\left(\frac{1}{\mu_\beta(B_R(x_0))}\int_{B_R(x_0)} |\nabla\phi|^2 \frac{\mathrm{d}x}{|x|^\beta}\right)^{\frac{1}{2}}, \qquad (0.0.29)$$

where $\overline{\phi_\gamma} = \mu_\gamma\left(B_R(x_0)\right)^{-1}\int_{B_R(x_0)} \phi|x|^{-\gamma}\,\mathrm{d}x$; the constant $P_{\gamma,\beta} > 0$ depends only on N, γ, β.

The above Poincaré inequality is a direct consequence of a more general one, called *Sobolev-Poincaré inequality* see [80, Chp. 15], and also [57, 94, 95, 96] and references therein for the known results. For the sake of completeness, we have decided to give a proof of the above inequality in Appendix-B.

In the proof of positivity estimates we will also use BMO -Bounded Mean Oscillation- weighted spaces and a weighted John-Nirenberg inequality; we recall here the definition and inequalities that we will use, for convenience of the reader.

Definition 0.0.12. *A function $f \in \mathrm{L}^1_{\gamma,\mathrm{loc}}(\Omega)$ is said to be in $BMO_\gamma(\Omega)$ if*

$$\|f\|_{BMO_\gamma(\Omega)} := \sup_{B\Subset\Omega}\frac{1}{\mu_\gamma(B)}\int_B |f - \overline{f}_B||x|^{-\gamma}\,\mathrm{d}x < +\infty,$$

where B are balls compactly contained in Ω and $\overline{f}_B = \mu_\gamma(B)^{-1}\int_B f|x|^{-\gamma}\,\mathrm{d}x$.

The following version of the John-Nirenberg Lemma can be found for instance in [80, Thm 18.3].

Lemma 0.0.13 (Weighted John-Nirenberg inequality, [80]). *Let $f \in BMO_\gamma(\Omega)$. Then, for any ball B compactly contained in Ω the following inequality holds*

$$\frac{1}{\mu_\gamma(B)}\int_B e^{\mathsf{s}|f-\overline{f}_B|}|x|^{-\gamma}\,\mathrm{d}x \leq \overline{\kappa}_5 \qquad \text{for any} \qquad 0 < \mathsf{s} \leq \frac{1}{\overline{\kappa}_6\|f\|_{BMO_\gamma(\Omega)}}, \qquad (0.0.30)$$

where $\overline{\kappa}_5$ and $\overline{\kappa}_6$ are positive constants depending only on N, γ.

From the above inequality there follows Corollary 0.0.14 that will play a crucial role in Chapter 2.

Corollary 0.0.14. *Let $u : \Omega \to \mathbb{R}$ be a positive measurable function such that $\log(u) \in BMO_\gamma(\Omega)$. Then the following inequality holds*

$$\|u\|_{\mathrm{L}^{\mathsf{s}}_\gamma(B)} \leq \overline{\kappa}_7^{\frac{2}{\mathsf{s}}}\,\mu_\gamma(B)^{\frac{2}{\mathsf{s}}}\,\|u\|_{\mathrm{L}^{-\mathsf{s}}_\gamma(B)} \qquad \text{for any} \qquad 0 < \mathsf{s} < \frac{1}{\overline{\kappa}_6\|\log(u)\|_{BMO_\gamma(\Omega)}}, \qquad (0.0.31)$$

where B is any ball compactly contained in Ω, $\overline{\kappa}_6 > 0$ is as in Lemma 0.0.13 and $\overline{\kappa}_7 > 0$ is a constant depending only on N, γ.

Proof. The proof of this result follows by the weighted John-Nirenberg inequality (0.0.30); it is a straightforward adaptation of the proof of [73, Theorem 4], see also [97, Proposition 4.4]. □

Chapter 1

Local upper bounds and energy estimates

The main result of this chapter is a precise and quantitative local upper bound, which ensures that sufficiently locally integrable solutions are indeed bounded at a later time, as precisely stated below; we state here the most general form of upper bounds, which includes the cases considered in Theorem 0.0.2.

Theorem 1.0.1. *Let u be a nonnegative local strong solution to WFDE on the cylinder $\Omega \times (0, T]$. Let moreover $p \geq 1$ if $m \in (m_c, 1)$ and $p > p_c$ if $m \in (0, m_c]$. Let $B_{2R_0}(x_0) \subset \Omega$, and assume that $B_{R_0}(x_0)$ satisfies either (a), (b) or (c) on page 26. Then there exist $\overline{\kappa}_8, \overline{\kappa}_9 > 0$ such that for any $0 < t - t_0 < T$*

$$\sup_{y \in B_{R_0}(x_0)} u\,(t, y) \leq \frac{\overline{\kappa}_8}{(t - t_0)^{(N-\gamma)\vartheta_p}} \left[\int_{B_{2R_0}(x_0)} |u(t_0, y)|^p \; |y|^{-\gamma} \; \mathrm{d}y \right]^{\sigma\vartheta_p} + \overline{\kappa}_9 \left[\frac{t - t_0}{R_0^\sigma} \right]^{\frac{1}{1-m}}, \quad (1.0.1)$$

where ϑ_p and σ are as in (0.0.21)*. The constants $\overline{\kappa}_8, \overline{\kappa}_9$ depend only on N, γ, β and on the quotient $|x_0|/R_0$; they both have an explicit expression given in* (1.4.1)*.*

We have already observed the main features of the above upper bounds, see Remark 0.0.3. Note that Theorem 0.0.2 is a particular case of the above, in the sense that the dependence on $|x_0|/R_0$ can be eliminated in the constants $\overline{\kappa}_8$ and $\overline{\kappa}_9$, simply by choosing the cases (1), (2) and (3) on page 26. We shall now proceed with the proof of the Theorem, in Subsection 1.4. Before that, we need a number of preliminary results, some of them having their own interest.

1.1 Local space-time energy estimates

We collect in this Subsection all the energy inequalities that we will use in the rest of Part I, the proof is quite technical; but follows by nowadays standard ideas; the hidden difficulty lies on the careful approximations needed to deal with the singular/degenerate nature of the weights. We postpone the proof to Appendix-A, in order to not to break the flow of the proof. In order to state the energy inequalities in all the possible scenarios, we introduce an auxiliary function: to avoid unnecessary complications, we consider balls $B_{R_1}(x_0) \subset B_{R_0}(x_0)$ such that $0 \notin \overline{B_{R_0}(x_0) \setminus B_{R_1}(x_0)}$.

Let $0 < R_1 < R_0$ and $\sigma = 2 + \beta - \gamma \in (0, \infty)$ and define

$$h_\sigma\left(R_0, R_1, x_0\right) := \begin{cases} \left(\dfrac{R_0 + |x_0|}{R_0 - R_1}\right)^{2-\sigma}, & \text{if } 0 < \sigma < 2\,, \\ 1 \vee \left(\dfrac{R_0 - R_1}{R_1 - |x_0|}\right)^{\sigma-2}, & \text{if } \sigma \geq 2 \text{ and } 0 \in B_{R_1}(x_0)\,, \\ 1 \vee \left(\dfrac{R_0 - R_1}{|x_0| - R_0}\right)^{\sigma-2}, & \text{if } \sigma \geq 2 \text{ and } 0 \notin \overline{B_{R_0}(x_0)}\,. \end{cases} \tag{1.1.1}$$

The function h_σ takes into account the change of geometry and covers more general cases than the ones defined in (1), (2) and (3) on page 26. We moreover observe that whenever B_R satisfies one of the hypothesis (1), (2) or (3) we have

$$h_\sigma\left(4R, R, x_0\right) \asymp h_\sigma\left(2R, R, x_0\right) \asymp C_{\gamma,\beta} \qquad \text{and} \qquad \overline{\kappa}_{17}^{-1} R^\sigma \leq R^2 \frac{\mu_\gamma(B_R(x_0))}{\mu_\beta(B_R(x_0))} \leq \overline{\kappa}_{17} R^\sigma\,,$$

where the constant $C_{\gamma,\beta} > 0$ depends only on the constants γ, β. This is consistent with the weighted estimates proved in the linear case, see [63, 72, 71] and also Remark 1.3.2 below.

Lemma 1.1.1 (Energy estimates)**.** *Let $x_0 \in \mathbb{R}^N$, $0 < R_1 < R$ such that $0 \notin \overline{B_R(x_0) \setminus B_{R_1}(x_0)}$ and let $0 < m < 1$, $0 \leq T_0 < T_1 < T$.*

- *Let u be a non-negative local strong subsolution to WFDE in $(T_0, T) \times B_R(x_0)$. Let $p > 1$ and assume $u \in \mathrm{L}^p_\gamma((T_0, T) \times B_R(x_0))$. Then there exists $c_1 > 0$ depending only on m, p, N, such that*

$$\begin{aligned} &\sup_{\tau \in [T_1, T]} \left\{ \int_{B_{R_1}(x_0)} u^p(\tau, x)\ |x|^{-\gamma}\, \mathrm{d}x \right\} + \int_{T_1}^T \int_{B_{R_1}(x_0)} \left| \nabla u^{\frac{p+m-1}{2}} \right|^2\ |x|^{-\beta}\, \mathrm{d}x\, \mathrm{d}t \\ &\leq c_1 \left[\frac{h_\sigma\left(R, R_1, x_0\right)}{(R - R_1)^\sigma} + \frac{1}{T_1 - T_0} \right] \int_{T_0}^T \int_{B_R(x_0)} \left(u^{p+m-1} + u^p\right)\ |x|^{-\gamma}\, \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{1.1.2}$$

- *Let $\delta > 0$ and $u \geq \delta$ be a local strong supersolution to WFDE in $(T_0, T) \times B_R(x_0)$.*
  - For all $0 < p < 1 - m$ there exists $c_2 > 0$ depending only on m, p, N, such that

$$\begin{aligned} &\int_{B_{R_1}(x_0)} u(T_0, x)^p |x|^{-\gamma}\, \mathrm{d}x + \int_{T_0}^{T_1} \int_{B_{R_1}(x_0)} \left| \nabla u^{\frac{p+m-1}{2}} \right|^2 |x|^{-\beta}\, \mathrm{d}x\, \mathrm{d}t \\ &\leq c_2 \left[\frac{h_\sigma(R, R_1, x_0)}{(R - R_1)^\sigma} + \frac{1}{T - T_1} \right] \int_{T_0}^T \int_{B_R(x_0)} (u^{p+m-1} + u^p) |x|^{-\gamma}\, \mathrm{d}x\, \mathrm{d}t; \end{aligned} \tag{1.1.3}$$

  - For all $p > 0$ there exists $c_3 > 0$ depending only on m, p, N, such that

$$\begin{aligned} &\sup_{\tau \in [T_1, T]} \left\{ \int_{B_{R_1}(x_0)} u(\tau, x)^{-p}\ |x|^{-\gamma}\, \mathrm{d}x \right\} + \int_{T_1}^T \int_{B_{R_1}(x_0)} \left| \nabla u^{\frac{-p+m-1}{2}} \right|^2\ |x|^{-\beta}\, \mathrm{d}x\, \mathrm{d}t \\ &\leq c_3 \left[\frac{h_\sigma(R, R_1, x_0)}{(R - R_1)^\sigma} + \frac{1}{T_1 - T_0} \right] \int_{T_0}^T \int_{B_R(x_0)} \left(u^{-p+m-1} + u^{-p}\right)\ |x|^{-\gamma}\, \mathrm{d}x\, \mathrm{d}t; \end{aligned} \tag{1.1.4}$$

The constants c_1, c_2, c_3 have an explicit expression given in the proofs.

The next Lemma plays the role of the celebrated Caccioppoli inequality and corresponds to the above estimates in the borderline case $p = 1 - m$. It will be a key ingredient in the proof of the positivity estimates of Chapter 2.

Lemma 1.1.2 (Caccioppoli estimates). *Let $m \in (0,1)$, $\delta > 0$ and $u \geq \delta$ be a local strong supersolution to WFDE in $(T_0, T) \times B_R(x_0)$. Let $\psi \in C_c^\infty(\Omega)$ with $\operatorname{supp}(\psi) \subseteq B_R(x_0) \subset \Omega$ and $T_0 \leq \tau < t \leq T$. We have*

$$\begin{aligned}
&\int_{B_R(x_0)} u(\tau,x)^{1-m}\psi^2\ |x|^{-\gamma}\,\mathrm{d}x + \frac{m^2}{2}(1-m)\int_\tau^t\int_{B_R(x_0)}\psi^2\ |\nabla \log u|^2\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq 2\,(1-m)\int_\tau^t\int_{B_R(x_0)}|\nabla\psi|^2\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t + \int_{B_R(x_0)} u(t,x)^{1-m}\ \psi^2\ |x|^{-\gamma}\,\mathrm{d}x\,.
\end{aligned} \tag{1.1.5}$$

We finally observe that letting $m \to 1^-$ in (1.1.5), and recalling that $u^{1-m}/(1-m) \to \log u$ in such limit, we recover the classical Caccioppoli estimate, valid for $m = 1$ and $\beta = \gamma = 0$, cf. [98, 73, 74, 46].

1.2 Behaviour of local L^p_γ norms.

Proposition 1.2.1. *Let $m \in (0,1)$ and $u\,(t,x) : [T_0, T_1] \times \Omega \to \mathbb{R}$ be a nonnegative local strong solution to the WFDE. Let $x_0 \in \Omega$ and $R > 0$ be such that $B_{2R}(x_0) \subset \Omega$. Then, for any $0 \leq t, \tau \in [T_0, T_1]$ we have*

$$\left[\int_{B_R(x_0)} u\,(t,x)\,\frac{\mathrm{d}x}{|x|^\gamma}\right]^{1-m} \leq \left[\int_{B_{2R}(x_0)} u\,(\tau,x)\,\frac{\mathrm{d}x}{|x|^\gamma}\right]^{1-m} + \overline{\kappa}_{10}\frac{\mu_\gamma(B_R(x_0))^{1-m}}{\rho_{x_0}^{\gamma,\beta}(R)}|t-\tau|, \tag{1.2.1}$$

where the constant $\overline{\kappa}_{10}$ depends only on N, m, γ and β. Moreover, under the assumption that $B_{R_0}(x_0)$ satisfies either (1), (2) or (3), (1.2.1) *becomes*

$$\left[\int_{B_R(x_0)} u\,(t,x)\,\frac{\mathrm{d}x}{|x|^\gamma}\right]^{1-m} \leq \left[\int_{B_{2R}(x_0)} u\,(\tau,x)\,\frac{\mathrm{d}x}{|x|^\gamma}\right]^{1-m} + \overline{\kappa}'_{10}\frac{\mu_\gamma(B_R(x_0))^{1-m}}{R^\sigma}|t-\tau|, \tag{1.2.2}$$

where $\overline{\kappa}'_{10} = \overline{\kappa}_{10}\,\overline{\kappa}_{17}\,\overline{\kappa}_{16}^{-1}$; $\overline{\kappa}_{16}\,, \overline{\kappa}_{17}$ depend only on N, γ, β and are as in (3.4.1), (0.0.20) *respectively.*

Remark. The above Lemma quantifies the displacement of local mass backward and forward in time, and has been first proved in [99] in the non-weighted case and for solutions to the Cauchy problem on $\mathbb{R}^N$: inequality (1.2.1) implies conservation of mass (letting $R \to \infty$, when $m > m_c$). It has also been used to prove estimates from below for the extinction time in different contexts: for the Cauchy problem when $m < m_c$, for any $m \in (0,1)$ for the Dirichlet problem or on Manifolds with negative curvature, see [52, 24, 48]. Also in this weighted case it allows one to prove the same results: we use it here (also) to prove lower bounds for the extinction time for the Minimal Dirichlet problem for all $m \in (0,1)$, see (2.0.1).

Proof. Let ϕ be a cut-off function supported in $B_{2R}(x_0)$ and let $\phi = 1$ in $B_R(x_0)$. In what follow we will write B_R instead of $B_R(x_0)$ when no confusion arises. We adopt the notation $\mathcal{L}_{\gamma,\beta} f =$

$|x|^{\gamma}\nabla\cdot\left(|x|^{-\beta}\nabla f\right)$, see also (3.4.7). Let us compute

$$\begin{aligned}\left|\frac{d}{dt}\int_{B_{2R}} u(t,x)\phi\left(x\right)|x|^{-\gamma}\,\mathrm{d}x\right| &= \left|\int_{B_{2R}} \mathcal{L}_{\gamma,\beta}\left(u^{m}\right)\phi\left(x\right)|x|^{-\gamma}\,\mathrm{d}x\right| \\ = \left|\int_{B_{2R}} u^{m}\mathcal{L}_{\gamma,\beta}\left(\phi\left(x\right)\right)|x|^{-\gamma}\,\mathrm{d}x\right| &\leq \int_{B_{2R}} u^{m}\left|\mathcal{L}_{\gamma,\beta}\left(\phi\left(x\right)\right)\right||x|^{-\gamma}\,\mathrm{d}x.\end{aligned} \tag{1.2.3}$$

Hölder's inequality with conjugate exponents $\frac{1}{m}$ and $\frac{1}{1-m}$ gives

$$\begin{aligned}\int_{B_{2R}} u^{m}\left|\mathcal{L}_{\gamma,\beta}\left(\phi\right)\right|\frac{\mathrm{d}x}{|x|^{\gamma}} &\leq \left[\int_{B_{2R}} u\phi\left(x\right)\frac{\mathrm{d}x}{|x|^{\gamma}}\right]^{m}\left[\int_{B_{2R}} \phi^{\frac{-m}{1-m}}\left|\mathcal{L}_{\gamma,\beta}\left(\phi\right)\right|^{\frac{1}{1-m}}\frac{\mathrm{d}x}{|x|^{\gamma}}\right]^{1-m} \\ &:= C\left(\phi\right)\left[\int_{B_{2R}} u\phi\left(x\right)\frac{\mathrm{d}x}{|x|^{\gamma}}\right]^{m}\end{aligned}$$

Notice that joining the above estimate and (1.2.3) we get the closed differential inequality

$$\left|\frac{d}{dt}\int_{B_{2R}} u\left(t,x\right)\phi\left(x\right)\frac{\mathrm{d}x}{|x|^{\gamma}}\right| \leq C(\phi)\left[\int_{B_{2R}} u\left(t,x\right)\phi\left(x\right)\frac{\mathrm{d}x}{|x|^{\gamma}}\right]^{m}.$$

An integration in time shows that for all $t,\tau\geq 0$ we have

$$\left(\int_{B_{2R}} u\left(t,x\right)\phi\left(x\right)\frac{\mathrm{d}x}{|x|^{\gamma}}\right)^{1-m} \leq \left(\int_{B_{2R}} u\left(\tau,x\right)\phi\left(x\right)\frac{\mathrm{d}x}{|x|^{\gamma}}\right)^{1-m} + (1-m)\,C\left(\phi\right)|t-\tau|\,.$$

Since ϕ is supported in B_{2R} and equal to 1 in B_R, this implies (1.2.1). The above proof is formal when considering weak or very weak solutions, in which case, it is quite lengthy (although standard) to make it rigorous: we start by considering the integrated version of inequality (1.2.3), which follows by Definition 0.0.1 of weak solution plus an integration by parts (that can be justified through approximation); we then conclude by a Grownwall-type argument.
The proof is concluded once we show that $C\left(\phi\right)$ is bounded: choosing ϕ as in Lemma 3.4.4 we get

$$\phi^{\frac{-m}{1-m}}\left|\mathcal{L}_{\gamma,\beta}\left(\phi\right)\right|^{\frac{1}{1-m}} \leq \overline{\kappa}_{10}\left(\rho_{x_0}^{\gamma,\beta}(R)\right)^{-\frac{1}{1-m}},$$

where the constant $\overline{\kappa}_{10}>0$ does not depend on x_0 but only on N,m,γ and β. Finally, we get

$$(1-m)\,C\left(\phi\right) = (1-m)\left[\int_{B_{2R}} \phi\left(x\right)^{\frac{-m}{1-m}}\left|\mathcal{L}_{\gamma,\beta}\left(\phi\left(x\right)\right)\right|^{\frac{1}{1-m}}\frac{\mathrm{d}x}{|x|^{\gamma}}\right]^{1-m} \leq \frac{\overline{\kappa}_{10}}{\rho_{x_0}^{\gamma,\beta}(R)}\mu_{\gamma}(B_R)^{1-m}.$$

Using (3.4.1) and (0.0.20) one easily gets (1.2.2). The proof is now concluded. □

When $p>1$ similar estimates still hold in the following slightly weaker form.

Proposition 1.2.2. *Let $m \in (0,1)$ and u be a nonnegative local strong solution to WFDE on $(0,T] \times \Omega$. Let $p > 1$ and $u(\tau) \in \mathrm{L}^p_{\gamma,\mathrm{loc}}(\Omega)$ for some $\tau \geq 0$. Let $x_0 \in \Omega$ and $0 < R_1 < R_0$ be such that $0 \notin \overline{B_{R_0}(x_0) \setminus B_{R_1}(x_0)}$. Then for any $t \geq \tau$ we have*

$$\left[\int\limits_{B_{R_1}(x_0)} u\,(t,x)^p \,\frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{1-m}{p}} \leq \left[\int\limits_{B_{R_0}(x_0)} u\,(\tau,x)^p \,\frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{1-m}{p}} + K_{R_0,R,p,\sigma,x_0}\,(t-\tau) \tag{1.2.4}$$

where the constant K_{R_0,R,p,σ,x_0} is given by

$$K_{R_0,R,p,\sigma,x_0} = c_p \frac{h_\sigma\,(R_0,R_1,x_0)}{(R_0-R_1)^\sigma}\left[\mu_\gamma\,(B_{R_0}\,(x_0) \setminus B_{R_1}\,(x_0))\right]^{\frac{1-m}{p}}, \tag{1.2.5}$$

where $c_p \sim m(1-m)/(p-1)$ depends only on p, m, N.

Remark. The above estimates prove stability of local L^p_γ norms. Analogous estimates have been proven in [24, 49, 78, 25] in the non-weighted case and in [52] on manifolds of non-positive curvature.

Proof. The energy inequality (3.3.4) can be written as follows, for any $0 \leq \psi \in C_c^\infty(\Omega)$

$$\frac{d}{dt}\int_\Omega u^p \psi^2 |x|^{-\gamma}\,\mathrm{d}x \leq \frac{2mp}{p-1}\int_\Omega u^{p+m-1}|\nabla\psi|^2\,|x|^{-\beta}\,\mathrm{d}x.$$

Using Hölder's inequality with exponents $p/(1-m)$ and $p/(p+m-1)$ we get

$$\begin{aligned}
&\int_\Omega u^{p+m-1}|\nabla\psi|^2\,|x|^{-\beta}\,\mathrm{d}x \\
&= \int_\Omega |\nabla\psi|^2|x|^{-\beta}|x|^{\frac{\gamma(p+m-1)}{p}}\psi^{\frac{-2(p+m-1)}{p}}\;u^{p+m-1}|x|^{\frac{-\gamma(p+m-1)}{p}}\psi^{\frac{2(p+m-1)}{p}}\,\mathrm{d}x \\
&\leq \left[\int_\Omega u^p\psi^2\,|x|^{-\gamma}\,\mathrm{d}x\right]^{1-\frac{1-m}{p}}\left[\int_\Omega |\nabla\psi|^{\frac{2p}{1-m}}|x|^{-\beta\frac{p}{1-m}}|x|^{\gamma\frac{p+m-1}{1-m}}\psi^{-2\frac{p+m-1}{1-m}}\,\mathrm{d}x\right]^{\frac{1-m}{p}}.
\end{aligned}$$

Combining the above estimates we obtain the following differential inequality

$$\frac{d}{dt}\int_\Omega u^p\psi^2|x|^{-\gamma}\,\mathrm{d}x \leq C_\psi\left[\int_\Omega u^p\psi^2\,|x|^{-\gamma}\,\mathrm{d}x\right]^{1-\frac{1-m}{p}},$$

which, integrated over (τ, t) gives us

$$\left[\int_\Omega u\,(t,x)^p\,\psi^2\,|x|^{-\gamma}\,\mathrm{d}x\right]^{\frac{1-m}{p}} \leq \left[\int_\Omega u\,(\tau,x)^p\,\psi^2\,|x|^{-\gamma}\,\mathrm{d}x\right]^{\frac{1-m}{p}} + \frac{(1-m)}{p}C_\psi\,(t-\tau)\,.$$

The above proof is formal when considering weak or very weak solutions: a rigorous derivation of the above inequality can be done in that case by using directly the energy inequality (3.3.4) and a Grownwall type argument. To conclude the proof, we need to show that C_ψ is finite. To this end, we choose $\psi = \phi^b$ with $0 \leq \phi \leq 1$ so that $\mathrm{supp}(\psi) \subseteq B_{R_0}$, $\mathrm{supp}(|\nabla\psi|) \subseteq B_{R_0}\,(x_0) \setminus B_{R_1}\,(x_0) := A_{R_0,R_1}$; we take $b > p/(1-m)$ so that

$$|\nabla\psi(x)|^{\frac{2p}{1-m}}\,\psi(x)^{-\frac{2(p+m-1)}{1-m}} \leq C_1\phi^{2b-\frac{2p}{1-m}}|\nabla\phi(x)|^{\frac{2p}{1-m}} \leq C_2(R_0-R_1)^{-\frac{2p}{1-m}}\,.$$

As a consequence, we obtain

$$\begin{aligned}
\frac{1-m}{p} C_\psi &= \frac{2m(1-m)}{p-1} \left[\int_{A_{R_0,R_1}} \frac{|\nabla \psi|^{\frac{2p}{1-m}}}{\psi^{\frac{2(p+m-1)}{1-m}}} |x|^{\gamma \frac{p+m-1}{1-m} - \beta \frac{p}{1-m}} \, \mathrm{d}x \right]^{\frac{1-m}{p}} \\
&\le \left[\frac{C_2^{\frac{p}{1-m}} c_{0,p}^{\frac{p}{1-m}}}{(R_0 - R_1)^{\frac{2p}{1-m}}} \int_{A_{R_0,R_1}} |x|^{(\gamma-\beta)\frac{p}{(1-m)}} \frac{\mathrm{d}x}{|x|^\gamma} \right]^{\frac{1-m}{p}} \\
&\le c_p \frac{h_\sigma(R_0, R_1, x_0)}{(R_0 - R_1)^\sigma} \mu_\gamma \left(A_{R_0,R_1} \right)^{\frac{1-m}{p}} := K_{R_0,R,p,\sigma,x_0} \,,
\end{aligned}$$

where $c_p, c_{0,p} \sim m(1-m)/(p-1)$, and in the last step we have used inequalities (3.3.11), (3.3.12), (3.3.13), depending on the different cases. □

1.3 Space-time smoothing effects for linear and nonlinear equations

We prove here a weighted space-time $\mathrm{L}^p_\gamma - \mathrm{L}^\infty$ smoothing effect, through a Moser-type iteration. This result represents the core of the proof of our main upper estimates, Theorem 1.0.1. Here we will cover more general cases than the ones defined in (1), (2) and (3).

Theorem 1.3.1. *Let $u \in \mathrm{L}^p_{\gamma,\mathrm{loc}}((0,T) \times B_R(x_0))$ be a nonnegative local strong (sub)solution to WFDE, let $p \ge 1$ if $m \in (m_c, 1)$ and $p > p_c$ if $m \in (0, m_c]$. Let $x_0 \in \mathbb{R}^N$, $0 < R_1 < R_0 < R$ be such that $0 \notin \overline{B_{R_0}(x_0) \setminus B_{R_1}(x_0)}$ and let $0 \le T_0 < T_1 < T$. Then there exists a constant $\overline{\kappa}_{11} > 0$ depending only on γ, β, N, m, p such that the following inequality holds*

$$\begin{aligned}
\sup_{(\tau,y) \in (T_1,T] \times B_{R_1}(x_0)} u(\tau, y) &\le \overline{\kappa}_{11} \left[\frac{h_\sigma(R_0, R_1, x_0)}{(R_0 - R_1)^\sigma} + \frac{1}{T_1 - T_0} \right]^{(N-\gamma+\sigma)\vartheta_p} \\
&\times \left[\int_{T_0}^T \int_{B_{R_0}(x_0)} (u^p + 1) \frac{\mathrm{d}x \, \mathrm{d}t}{|x|^\gamma} \right]^{\sigma \vartheta_p}
\end{aligned} \tag{1.3.1}$$

where σ and ϑ_p are defined in (0.0.21), *h_σ is defined in* (1.1.1) *and $\overline{\kappa}_{11}$ is given in* (1.3.19).

Remark 1.3.2. This result is similar to Theorem 2.4 of [24], but in this weighted case an important geometric factor appears: h_σ defined in (1.1.1). Since the weights are not translation invariant, the factor h_σ will change in a strong way the behaviour of the local estimates in the different situations (a), (b), (c) and (d), described in Section 0.0.1. The technical hypothesis $0 \notin \overline{B_{R_0}(x_0) \setminus B_{R_1}(x_0)}$ assumed in Theorem 1.3.1 gurantees that the quantity $h_\sigma(R_0, R_1, x_0)$ is finite. In the non-weighted case, similar estimates are proven in [49, 25, 76], with different operators and nonlinearities.

Remark 1.3.3 (The linear case with coefficients)**.** A close inspection of the proofs (both of the above Theorem and of the energy inequalities) reveals that indeed the above result still holds in the limit $m \to 1^-$, and even for more general equations, see Proposition 3.1.4; roughly speaking, we can consider solutions v to the linear equation $v_t = \mathcal{L}_{\gamma,\beta}\, v$, where the prototype operator has the form $\mathcal{L}_{\gamma,\beta}\, v = |x|^\gamma \nabla \cdot \left(a(t,x) |x|^{-\beta} \nabla v \right)$ with $0 < \lambda_0 \le a(t,x) \le \lambda_1 < \infty$. We refer to Subsection 3.1 for more details.

We now proceed with the proof of Theorem 1.3.1, which relies on a variant of the celebrated Moser iteration, adapting the proof of Theorem 2.4 of [24] to the weighted setting under consideration, for this reason we will be rather sketchy in the proofs. As already mentioned in Subsection 0.0.1, the role of weighted Sobolev inequalities will be played here by the Caffarelli-Kohn-Nirenberg inequalities (CKNI2), in the following form:

Lemma 1.3.4 (Iterative version of CKNI Inequality). *Let $r^* := 2(N-\gamma)/(N-2-\beta)$ with γ, β as in* (0.0.11). *Then for any ball $B_R(x_0)$ and for any $f \in \mathrm{L}^2((T_0,T_1); H^1_{\gamma,\beta}B_R(x_0))$ and for any $a \in [1, r^*/2]$ the following inequality holds*

$$\begin{aligned}\int_{T_0}^{T_1}\int_{B_R(x_0)} f^{2a}\frac{\mathrm{d}y\,\mathrm{d}t}{|y|^\gamma} &\le 2\,S^2_{\gamma,\beta}\left[\int_{T_0}^{T_1}\int_{B_R(x_0)} f^2\frac{\mathrm{d}y\,\mathrm{d}t}{|y|^\gamma} + \mu_\gamma(B_R(x_0))^{\frac{\sigma}{N-\gamma}}\int_{T_0}^{T_1}\int_{B_R(x_0)} |\nabla f|^2\frac{\mathrm{d}y\,\mathrm{d}t}{|y|^\beta}\right] \\ &\times \sup_{t\in[T_0,T_1]}\left(\mu_\gamma(B_R(x_0))^{-1}\int_{B_R(x_0)} f^{2(a-1)q}(y,t)\,\frac{\mathrm{d}y}{|y|^\gamma}\right)^{\frac{1}{q}},\end{aligned} \tag{1.3.2}$$

where $q = r^*/(r^*-2) = (N-\gamma)/\sigma$, σ is given in (0.0.21) *and the constant $S_{\gamma,\beta} > 0$ is the one appearing in CKNI2.*

Proof. We will write B_R instead of $B_R(x_0)$ and prove the result on $(T_0,T_1)\times B_R$. Since $r^*/2$ and q are conjugate Hölder exponents, using Hölder inequality and (CKNI2) we get

$$\begin{aligned}\int_{B_R(x_0)} f^{2a}\frac{\mathrm{d}x}{|x|^\gamma} &= \int_{B_R(x_0)} f^2 f^{2(a-1)}\frac{\mathrm{d}x}{|x|^\gamma} \le \left[\int_{B_R(x_0)} f^{r^*}\frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{2}{r^*}}\left[\int_{B_R(x_0)} f^{2(a-1)q}\frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{1}{q}} \\ &\le 2S^2_{\gamma,\beta}\left[\mu_\gamma(B_R(x_0))^{\frac{-\sigma}{N-\gamma}}\int_{B_R(x_0)} f^2\frac{\mathrm{d}x}{|x|^\gamma} + \int_{B_R(x_0)}|\nabla f|^2\,\frac{\mathrm{d}x}{|x|^\beta}\right]\sup_{t\in(T_0,T_1)}\left[\int_{B_R(x_0)} f^{2(a-1)q}(x,t)\frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{1}{q}}.\end{aligned}$$

Inequality (1.3.2) follows by integrating in time. □

Proof of Theorem 1.3.1. Throughout this proof u will be a local strong solution to WFDE defined on $Q = (T_0, T] \times B_R(x_0)$. We shall define $v(t,x) = u(t,x) \vee 1$, so that v is a local strong *subsolution* to the same equation. Notice that v satisfies $u \le v \le u+1$ almost everywhere in Q. Let us fix $x_0 \in \mathbb{R}^N$, and simply denote $B_R = B_R(x_0)$ ($B_{R_1} = B_{R_1}(x_0)$ resp.) when there is no ambiguity. In what follows we will make some a priori estimates of the solution: the quantity h_σ will be involved and we can assume it to be bounded, as it will be clear later. We recall that $\sigma = 2+\beta-\gamma > 0$, $r^* = 2(N-\gamma)/(N-2-\beta)$, and we set $q = (N-\gamma)/\sigma$. We will split the proof into several steps. We first deal with the case $p > 1$, the case $p = 1$, which only affects the good fast diffusion range $m_c < m < 1$, requires some extra work, and will be discussed at the last Step.

• Step 1. *Preparation of the iteration step.* Let $0 < R_1 < R_0 < R$ and $0 \le T_0 < T_1 < T$ and define $Q_0 := (T_0, T] \times B_{R_0}(x_0)$ and $Q_1 := (T_1, T] \times B_{R_1}(x_0)$. Recall that we are assuming $p > 1$. We are going to prove the following inequality:

$$\begin{aligned}\iint_{Q_1} v^{a(p+m-1)}\frac{\mathrm{d}t\,\mathrm{d}x}{|x|^\gamma} &\le (2S_{\gamma,\beta})^2\,(2c_1)^{1+\frac{1}{q}}\left[\frac{h_\sigma(R_0,R_1,x_0)}{(R_0-R_1)^\sigma} + \frac{1}{T_1-T_0}\right]^{1+\frac{1}{q}} \\ &\times\left[\iint_{Q_0} v^p\,\frac{\mathrm{d}t\,\mathrm{d}x}{|x|^\gamma}\right]^{1+\frac{1}{q}},\end{aligned} \tag{1.3.3}$$

where $q = r^*/(r^*-2)$, $a \in (1, r^*/2)$ is such that $(p+m-1)(a-1)\,q = p$; moreover, $c_1 > 0$ only depends on m, p, N and is given in the energy inequality (1.1.2).

To prove the above inequality, we first recall the CKNI inequality (1.3.2) with $f^2 = v^{p+m-1}$:

$$\iint_{Q_1} v^{a(p+m-1)} \frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma} \le 2\, S^2_{\gamma,\beta} \left[\iint_{Q_1} \left(v^{p+m-1}|x|^{-\gamma} + \mu_\gamma(B_{R_1})^{\frac{\sigma}{N-\gamma}} |\nabla v^{\frac{p+m-1}{2}}|^2 |x|^{-\beta}\right) \mathrm{d}x\,\mathrm{d}t\right]$$
$$\times \left[\sup_{t\in(T_1,T]} \mu_\gamma(B_{R_1})^{-1} \int_{B_{R_1}(x_0)} v^{(p+m-1)(a-1)q} |x|^{-\gamma}\,\mathrm{d}x\right]^{\frac{1}{q}}. \tag{1.3.4}$$

Next, we estimate the first term in the right-hand side using the upper energy inequality (1.1.2) applied to $v \ge 1$, so that $v^{p+m-1} \le v^p$ and

$$\iint_{Q_1} \left(v^{p+m-1}\,|x|^{-\gamma} + \mu_\gamma(B_{R_1})^{\frac{\sigma}{N-\gamma}}\,|\nabla v^{\frac{p+m-1}{2}}|^2\,|x|^{-\beta}\right) \mathrm{d}x\,\mathrm{d}t \le \tilde{J} \iint_{Q_0} v^p\,|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t, \tag{1.3.5}$$

where we have assumed (without loss of generality[1]) that

$$\tilde{J} := 2c_1 \mu_\gamma(B_{R_1})^{\frac{\sigma}{N-\gamma}} \left[\frac{h_\sigma(R_0,R_1,x_0)}{(R_0-R_1)^\sigma} + \frac{1}{T_1-T_0}\right] \ge 1. \tag{1.3.6}$$

We next estimate the second term in the right-hand side of (1.3.4), using again (1.1.2) applied to $v \ge 1$. Since we are assuming $p > p_c$, we can choose $a \in (1, r^*/2)$ such that $(p+m-1)(a-1)q = p$, to get

$$\left[\sup_{t\in(T_1,T]} \int_{B_{R_1}(x_0)} v^{(p+m-1)(a-1)q} \frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{1}{q}} \tag{1.3.7}$$
$$\le \left[2c_1 \left(\frac{h_\sigma(R_0,R_1,x_0)}{(R_0-R_1)^\sigma} + \frac{1}{T_1-T_0}\right) \iint_{Q_0} v^p \frac{\mathrm{d}t\,\mathrm{d}x}{|x|^\gamma}\right]^{\frac{1}{q}}.$$

Combining inequalities (1.3.4), (1.3.5) and (1.3.7) we finally obtain (1.3.3).

• Step 2. *The k^{th} iteration step.* We define a sequence of increasing exponents $p_k \to +\infty$ and nested cylinders $Q_k \subset Q_{k+1}$ as follows. We define first the exponents p_k, recalling that $q = r^*/(r^*-2)$:

$$p_{k+1} = \left(1+\frac{1}{q}\right) p_k + m - 1 = \left(1+\frac{1}{q}\right)^{k+1} [p_0 - q(1-m)] + q(1-m)\,. \tag{1.3.8}$$

Notice that in the range $m \in (0, m_c]$, we have $p_k < p_{k+1} \to +\infty$ if and only if $p_0 > p_c$, while when $m \in (m_c, 1)$ it suffices to have $p_0 \ge 1$; this justifies the assumption on the initial datum.
Next, we define the cylinders $Q_k \subset Q = (T_0, T] \times B_{R_0}$ as follows:

$$\begin{aligned} Q_k &:= (T_k, T] \times B_{R_k}(x_0) \qquad \text{such that} \\ Q &= Q_0 \supset Q_k \supset Q_{k+1} \to Q_\infty = (T_\infty, T] \times B_{R_\infty} \end{aligned} \tag{1.3.9}$$

[1] Indeed, by (1.3.10), it is clear that this is true at the k-th iteration step for large k. Indeed we could have done everything by replacing k with $k + a_0$ for a suitable large a_0: after the whole iteration process, this would affect only estimate (1.3.19), where C_1 and C_2 would depend on a_0, but a posteriori the dependence on a_0 can be easily eliminated. We have decided to omit this, to focus on the main ideas.

where we have chosen a decreasing sequence of radii $R_0 > R_k > R_{k+1} \to R_\infty$ and an increasing sequence of times $T_0 < T_k < T_{k+1} \to T_\infty$ such that

$$R_k - R_{k+1} = C_1 \frac{R_0 - R_\infty}{(k+1)^\alpha} \qquad \text{and} \qquad T_{k+1} - T_k = C_2 \frac{T_\infty - T_0}{(k+1)^{\alpha\sigma}}\,, \tag{1.3.10}$$

where (note that we choose α so that C_1, C_2 will be finite)

$$\alpha = 2 \vee \frac{1}{\sigma}, \qquad C_1 = \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)^\alpha}\right)^{-1} \qquad \text{and} \qquad C_2 = \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha\sigma}}\right)^{-1}. \tag{1.3.11}$$

Plugging all the above defined quantities in inequality (1.3.3), noting that $p = p_k$ implies $p_{k+1} = a(p+m-1)$, we can write the k^{th} iteration step as follows

$$\begin{aligned}
\left[\iint_{Q_{k+1}} v^{p_{k+1}} |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_{k+1}}} &\le [2S_{\gamma,\beta}]^{\frac{2}{p_{k+1}}}\, [2c_1]^{\left(1+\frac{1}{q}\right)\frac{1}{p_{k+1}}} \\
&\quad\times \left[\frac{h_\sigma\left(R_k, R_{k+1}, x_0\right)}{\left(R_k - R_{k+1}\right)^\sigma} + \frac{1}{T_{k+1} - T_k}\right]^{\left(1+\frac{1}{q}\right)\frac{1}{p_{k+1}}} \\
&\quad\times \left[\iint_{Q_k} v^{p_k} |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{p_k}{p_{k+1}}\left(1+\frac{1}{q}\right)\frac{1}{p_k}}.
\end{aligned} \tag{1.3.12}$$

Bounds for the constants. It is convenient to bound the constants appearing in (1.3.12) by a quantity which does not depend on p, but only on $m, \gamma, \beta, R_\infty$ and R_0. Recall the expression of c_1, given in (1.1.2)

$$c_1 = 2K_\psi\, c_{m,p_k}^{-1} \qquad \text{with} \qquad c_{m,p_k} = \frac{p_k - 1}{p_k} \wedge \frac{2m\left(p_k - 1\right)^2}{\left(p_k + m - 1\right)^2}\,,$$

and with $K_\psi > 0$ depending only on N. The quantity c_{m,p_k} needs to be bounded uniformly for all $k \ge 0$: since $p_k > p_0 > 1 \vee p_c$ it is easy to show that

$$c_{m,p_k} \ge \left(1 - \frac{1}{p_0}\right) \wedge 2m\left(1 - \frac{m}{p_0 + m - 1}\right)^2 \quad \text{so that} \quad c_1 = c_1(p_k) \le \overline{c} = \overline{c}(m, N, p_0) < +\infty\,.$$

As a consequence, $c_1(p_k) \le \overline{c}$ can be bounded uniformly by a constant that depends only on N, m, p_0.

On the other hand, $h_\sigma(R_k, R_{k+1}, x_0)$ may be bounded by a fixed quantity depending only on σ, R_0 and R_∞,

$$h_\sigma\left(R_k, R_{k+1}, x_0\right) \le h_\sigma\left(R_0, R_\infty, x_0\right)(k+1)^{\alpha(2-\sigma)_+}\, C_1^{-(2-\sigma)_+}, \tag{1.3.13}$$

the constant C_1 being the one appearing in (1.3.11).
Finally, we can rewrite the k^{th} iteration step (1.3.12) as follows

$$\left[\iint_{Q_{k+1}} v^{p_{k+1}}\, |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_{k+1}}} \le I_{k+1}^{\frac{1}{p_{k+1}}} \left[\iint_{Q_k} v^{p_k} |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_{k+1}}\left(1+\frac{1}{q}\right)}, \tag{1.3.14}$$

where the constant I_{k+1} is bounded by,

$$\begin{aligned}
I_{k+1} &\le [2S_{\gamma,\beta}]^2 \left[2\overline{c}\left(\frac{h_\sigma\left(R_0, R_\infty, x_0\right) C_1^{-\frac{b}{\alpha}}}{\left(R_0 - R_\infty\right)^\sigma} + \frac{C_2^{-1}}{T_\infty - T_0}\right)\right]^{1+\frac{1}{q}} (k+1)^{b\left(1+\frac{1}{q}\right)} \\
&:= J_0\, J_1^{1+\frac{1}{q}}\, (k+1)^{b\left(1+\frac{1}{q}\right)}\,.
\end{aligned} \tag{1.3.15}$$

where $b = 2\alpha$ if $0 < \sigma < 2$, $b = \sigma\alpha$ otherwise. The assumption (1.3.11) on α ensures that $b > 1$.

• STEP 3. *The iteration.* We now iterate the inequalities (1.3.14) and obtain

$$\begin{aligned}\left[\iint_{Q_{k+1}} v^{p_{k+1}}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_{k+1}}} &\le \left[\prod_{j=1}^{k+1} I_j^{\frac{1}{p_{k+1}}\left(1+\frac{1}{q}\right)^{k+1-j}}\right] \\ &\quad\times\left[\iint_{Q_0} v^{p_0}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_{k+1}}\left(1+\frac{1}{q}\right)^{k+1}}.\end{aligned} \tag{1.3.16}$$

Using inequality (1.3.15) we can estimate the first term appearing in right-hand side of (1.3.16):

$$\begin{aligned}\prod_{j=1}^{k+1} I_j^{\frac{1}{p_{k+1}}\left(1+\frac{1}{q}\right)^{k+1-j}} &\le \left[J_0 J_1^{1+\frac{1}{q}}\right]^{\frac{1}{p_{k+1}}\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^{j}}\left[(k+1)^{b'}\right]^{\frac{1}{p_{k+1}}} \\ &\quad\times\left[k^{b'}\right]^{\frac{1}{p_{k+1}}\left(1+\frac{1}{q}\right)}\cdots\left[2^{b'}\right]^{\frac{1}{p_{k+1}}\left(1+\frac{1}{q}\right)^{k-1}} \\ &= \left[J_0 J_1^{1+\frac{1}{q}}\right]^{\frac{1}{p_{k+1}}\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^{j}}\prod_{j=1}^{k+1} j^{\frac{b'}{p_{k+1}}\left(1+\frac{1}{q}\right)^{k+1-j}},\end{aligned}$$

where $b' = b\left(1+\frac{1}{q}\right)$. Notice that there is a constant $c' > 0$ depending on p_0, N and γ, β such that

$$\begin{aligned}\lim_{k\to\infty}\prod_{j=1}^{k+1} j^{\frac{b'}{p_{k+1}}\left(1+\frac{1}{q}\right)^{k+1-j}} &\le \exp\left(\frac{b'}{p_0 - q\,(1-m)}\sum_{j=1}^{\infty}\left(\frac{q}{q+1}\right)^{j}\log j\right) \\ &\le (c')^{\frac{q}{p_0-q(1-m)}} < +\infty\,.\end{aligned}$$

Using the expression (1.3.8) of p_k, we see that

$$\lim_{k\to\infty}\frac{\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^{j}}{p_{k+1}} = \frac{q}{p_0 - q\,(1-m)} \qquad\text{and}\qquad \lim_{k\to\infty}\frac{\left(1+\frac{1}{q}\right)^{k+1}}{p_{k+1}} = \frac{1}{p_0 - q\,(1-m)}.$$

We can now take the limit in (1.3.16) as $k \to \infty$ obtaining (recall that $\lim_{k\to\infty}\|f\|_{\mathrm{L}^{p_k}_{\gamma}(Q_k)} \ge \|f\|_{\mathrm{L}^\infty(Q_\infty)}$)

$$\|v\|_{\mathrm{L}^\infty(Q_\infty)} \le (c'\,J_0)^{\frac{q}{p_0-q(1-m)}}\,J_1^{\frac{q+1}{p_0-q(1-m)}}\left(\iint_{Q_0} v^{p_0}\,|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_0-q(1-m)}}. \tag{1.3.17}$$

Recalling that J_0 and J_1 are as in (1.3.15), and that $v(t,x) = u(t,x)\vee 1$, so that $u^{p_0} \le v^{p_0} \le u^{p_0}+1$, we obtain from (1.3.17)

$$\sup_{Q_\infty} u \le \overline{\kappa}_{11}\left[\frac{h_\sigma(R_0,R_\infty,x_0)}{(R_0-R_\infty)^\sigma} + \frac{1}{T_\infty - T_0}\right]^{\frac{q+1}{p_0-q(1-m)}}\left[\iint_{Q_0}(u^{p_0}+1)\,\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right]^{\frac{1}{p_0-q(1-m)}}, \tag{1.3.18}$$

where $\overline{\kappa}_{11} > 0$ depends only on m, p_0, N, γ and β. We see that this is exactly inequality (1.3.1), recalling that $r^* := 2(N-\gamma)/(N-2-\beta)$, $q = r^*/(r^*-2)$, $\sigma\vartheta_{p_0} = 1/(p_0 - q(1-m))$ and $(N-\gamma+\sigma)\vartheta_{p_0} = \frac{q+1}{p_0-q(1-m)}$ and that

$$\overline{\kappa}_{11} = [2S_{\gamma,\beta}]^{\frac{2q}{p_0-q(1-m)}} \left[2\overline{c}\,(C_1^{-\frac{b}{\alpha}} \vee C_2^{-1})\right]^{\frac{q+1}{p_0-q(1-m)}} \tag{1.3.19}$$

where $S_{\gamma,\beta}$ is as in Proposition (0.0.10), C_1, C_2 are as in (1.3.11), $b = 2\alpha$ if $0 < \sigma < 2$, $b = \sigma\alpha$ otherwise, and $\overline{c} = \overline{c}(m, N, p_0)$, is as in Step 2. The proof of Theorem 1.3.1 in the case $p_0 > 1$ is concluded by letting $p_0 = p$, $R_\infty = R_1$ and $T_\infty = T_1$.

• STEP 4. *The case* $p = 1$. So far, we have proven the space time smoothing effect for solutions in $\mathrm{L}^{p_0}_\gamma$ for any $p_0 > 1$. Unfortunately we cannot simply take the limit as $p_0 \to 1$ in inequality (1.3.17) since the constant c' would blow up, being proportional to $c_1 \sim (p_0-1)^{-1}$, explicitly given in (1.1.2). We show how to deal with the limiting case $p = 1$. A standard way to proceed is to first prove the result for bounded initial data, for instance $u_{0,n} = u_0 \wedge n$, then by a lengthy but straightforward approximation procedure the result holds for L^1_γ solutions. We are going to use inequality (1.3.17) which holds true on any couple of cylinders of the form $Q_\infty \subset \underline{Q} := (\underline{T}, T] \times B_{\underline{R}}(x_0) \subset \overline{Q} := (\overline{T}, T] \times B_{\overline{R}}(x_0) \subset Q_0$ for any $p_0 > 1$, and implies (by Hölder's and Young's inequalities)

$$\|v\|_{\mathrm{L}^\infty(\underline{Q})} \le \frac{1}{2}\|v\|_{\mathrm{L}^\infty(\overline{Q})} + c'' J_1^{\frac{q+1}{1-q(1-m)}} \|v\|^{\sigma\vartheta_1}_{\mathrm{L}^1(\overline{Q})} \tag{1.3.20}$$

where $J_1 > 0$ is as in (1.3.15) and can be estimated as follows,

$$J_1 \le \overline{c}' \left[\frac{h_\sigma\left(\overline{R}, \underline{R}, x_0\right)}{\left(\overline{R}-\underline{R}\right)^\sigma} + \frac{1}{\underline{T}-\overline{T}}\right] \tag{1.3.21}$$

with $\overline{c}' = 2\,\overline{c}\,C_1^{-b/\alpha} \vee C_2^{-1}$, with $\overline{c}', C_1, C_2 > 0$, as in the previous step. Moreover,

$$c'' = 2^{\frac{\sigma(p-1)}{\sigma-(N-\gamma)(1-m)}} (c'\,J_0)^{\frac{q}{1-q(1-m)}} > 0 \tag{1.3.22}$$

and $c', J_0 > 0$ are as in the previous step, and depend only on p_0, N and γ, β.
We are now in the position to iterate the above inequality, using ideas inspired by a classical Lemma due to DeGiorgi, that can be found in many different sources, for instance see Lemma 3.6 of [97]. Fix $0 < \tau < 1$ and define cylinders $Q_i := (t_i, T] \times B_{r_i}(x_0)$ where

$$\begin{aligned} r_0 &= \underline{R} \quad \text{and} \quad r_{i+1} := r_i + (1-\tau)\tau^i(\overline{R}-\underline{R}) \\ t_0 &= \underline{T} \quad \text{and} \quad t_{i+1} := t_i - (1-\tau^\sigma)\tau^{i\sigma}(\underline{T}-\overline{T})\,. \end{aligned}$$

We iterate (1.3.20) as follows

$$\begin{aligned} \|v\|_{\mathrm{L}^\infty(Q_0)} &\le \frac{1}{2}\|v\|_{\mathrm{L}^\infty(Q_1)} + c''' \left[\frac{h_\sigma\left(\overline{R}, \underline{R}, x_0\right)}{\left(\overline{R}-\underline{R}\right)^\sigma} + \frac{1}{\underline{T}-\overline{T}}\right]^{(q+1)\vartheta_1} \|v\|^{\sigma\vartheta_1}_{\mathrm{L}^1(\overline{Q})} \tau^{-i\sigma(q+1)\vartheta_1} \\ &\le \left(\frac{1}{2}\right)^k \|v\|_{\mathrm{L}^\infty(Q_k)} + \\ &\quad + c''' \left[\frac{h_\sigma\left(\overline{R}, \underline{R}, x_0\right)}{\left(\overline{R}-\underline{R}\right)^\sigma} + \frac{1}{\underline{T}-\overline{T}}\right]^{(q+1)\vartheta_1} \|v\|^{\sigma\vartheta_1}_{\mathrm{L}^1(\overline{Q})} \sum_{i=0}^{k-1} \left(2\tau^{\sigma(q+1)\vartheta_1}\right)^{-i} \end{aligned} \tag{1.3.23}$$

where $c''' := c''\left(\frac{\overline{c}'}{[(1-\tau^\sigma)\wedge(1-\tau)^\sigma]}\right)^{(q+1)\vartheta_1}$. Taking the limit $k \to \infty$ we get

$$\|v\|_{\mathrm{L}^\infty(Q_0)} = \|v\|_{\mathrm{L}^\infty(\underline{Q})} \le \overline{\kappa}_{11}\left[\frac{h_\sigma\left(\overline{R},\underline{R},x_0\right)}{\left(\overline{R}-\underline{R}\right)^\sigma} + \frac{1}{\underline{T}-\overline{T}}\right]^{(q+1)\vartheta_1} \|v\|^{\sigma\vartheta_1}_{\mathrm{L}^1(\overline{Q})} \tag{1.3.24}$$

Finally, the constant $\overline{\kappa}_{11} = c'''\sum_{k=1}^\infty \left(2\tau^{\sigma(q+1)\vartheta_1}\right)^{-i} < \infty$, whenever $2^{-\sigma(q+1)\vartheta_1} < \tau < 1$. We have proven inequality (1.3.18) also when $u_0 \in \mathrm{L}^1_\gamma$, and the only thing that changes is the constant $\overline{\kappa}_{11}$, which in any case only depends on p_0, N and γ, β and we can even fix $p_0 > 1$ taking for instance $p_0 = 2$. $\square$

1.4 Proof of the main Theorem

The aim of this section is to prove Theorem 1.0.1: we only sketch the main points, which are analogous to those in [24, Section 2.3], where a detailed proof is given, just for simplicity here we take $t_0 = 3^{-\sigma}t$. Let $u(t,x)$ be a weak solution to WFDE in the cylinder $(0,T)\times B_{2R_0}(x_0)$, and let $0<\tau<T$, and define $\varepsilon = 1/3$ and $\rho = (3/2)R_0$. We apply Theorem 1.3.1 to the rescaled function

$$\hat{u}\left(t,x\right) = Mu\left(\tau t,\rho x\right) \qquad \text{where} \qquad M = \left(\frac{\rho^\sigma}{\tau}\right)^{\frac{1}{1-m}},$$

which turns out to be a solution to the WFDE over the cylinder $Q = (0,1]\times B_{(1+\varepsilon)}\left(\rho^{-1}x_0\right)$. Consider the cylinders $Q_0 = (0,1]\times B_1\left(\rho^{-1}x_0\right)$ and $Q_1 = ((1/3)^\sigma,1]\times B_{1-\varepsilon}\left(\rho^{-1}x_0\right)$. Applying estimate (1.3.1) to $\hat{u}$ over these two cylinders we get

$$\sup_{Q_1}\hat{u} \le \overline{\kappa}_{11}\left[2\,3^\sigma h_\sigma\left((3/2)R_0,R_0,x_0\right)\right]^{\frac{q+1}{p-q(1-m)}}\left[\iint_{Q_0}\left(\hat{u}^p+1\right)\,|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p-q(m-1)}}.$$

Applying the inequalities obtained in Proposition 1.2.1 and Proposition 1.2.2 to $\hat{u}$, on the domains $B_1\left(\rho^{-1}x_0\right) \subset B_{1+\varepsilon}\left(\rho^{-1}x_0\right)$, for times $t\in[0,1]$ and integrating them in t over $(0,1)$ we obtain

$$\int_0^1\int_{B_1}\hat{u}^p\,|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t \le 2^{\frac{p+m-1}{1-m}}\int_{B_{1+\varepsilon}}\hat{u}\left(0,x\right)^p\,|x|^{-\gamma}\,\mathrm{d}x + \frac{(1-m)\,2^{\frac{p+m-1}{1-m}}}{p+1-m}\,\mathcal{K},$$

which holds for any $p>p_c$ if $m\in(0,m_c]$ or $p\ge 1$ if $m\in(m_c,1)$; notice that $\mathcal{K}>0$ is an upper bound for the two constants given in (1.2.1) and (1.2.4) when $p=1$ or $p>1$ respectively; the expression of $\mathcal{K}$ will be given below in (1.4.1). Rescaling back to u, using inequality $(a+b)^s \le k_1a^s + k_2b^s$ and recalling that $h_\sigma\left(1,1-\varepsilon,\rho^{-1}x_0\right) = h_\sigma\left(\rho,\rho(1-\varepsilon),x_0\right)$ and $\mu_\gamma(B_1(\rho^{-1}x_0) = \rho^{\gamma-N}\mu_\gamma(B_\rho(x_0))$, we finally obtain

$$\sup_{((1/3)^\sigma\tau,\tau]\times B_{R_0}(x_0)} u \le \frac{C_1}{\tau^{\frac{q}{p+q(m-1)}}}\left[\int_{B_{2R_0}(x_0)} u\left(0,y\right)^p|y|^{-\gamma}\,\mathrm{d}y\right]^{\frac{1}{p-q(m-1)}} + C_2\left[\frac{\tau}{R_0^\sigma}\right]^{\frac{1}{1-m}},$$

where the constants are (recall that $\varepsilon = 1/3$ and $\rho = (3/2)R_0$):

$$
\begin{aligned}
C_1 &= k_1 \overline{\kappa}_{11} 2^{\frac{p+m-1}{(1-m)(p+q(m-1))}} \left[2 \cdot 3^\sigma h_\sigma\left((3/2)R_0, R_0, x_0\right)\right]^{\frac{(q+1)}{p-q(1-m)}}, \\
C_2 &= k_2 \overline{\kappa}_{11} \left[\frac{(1-m)\, 2^{\frac{p+m-1}{1-m}}}{p+1-m} \mathcal{K} + \frac{\mu_\gamma(B_{(3/2)R_0}(x_0))}{((3/2)R_0)^{N-\gamma}}\right]^{\frac{1}{p+q(m-1)}} \\
&\quad \times \left[2\, 3^\sigma h_\sigma\left((3/2)R_0, R_0, x_0\right)\right]^{\frac{(q+1)}{p-q(1-m)}}, \\
\mathcal{K} &= \begin{cases} c_p \left(3^\sigma h_\sigma\left(2R_0, (3/2)R_0, x_0\right)\right)^{\frac{p}{1-m}} (2R_0)^{\gamma-N} \mu_\gamma\left(B_{2R_0}(x_0)\right) & \text{if } p > 1\ , \\ \overline{\kappa}_{10}^{\frac{1}{1-m}} (2R_0)^{\gamma-N} \mu_\gamma\left(B_{2R_0}(x_0)\right) \left(\rho_{x_0}^{\gamma,\beta}(R_0/3)\right)^{-1} & \text{if } p = 1, \end{cases}
\end{aligned}
\tag{1.4.1}
$$

where c_p is as in (1.2.5). This concludes the proof. $\square$

Chapter 2

Positivity estimates

This Chapter is devoted to the proof of our main positivity result, Theorem 0.0.4. The proof is delicate, quite long and technical, and represents the major novelty of this work, as already explained. The strategy of the proof of our local lower bounds, Theorem 0.0.4, relies on the study of the worst-case scenario: we will prove lower bounds for solutions to a "smaller problem", that we will call the *Minimal Dirichlet Problem* (MDP), following [24]. Then, by nowadays standard comparison arguments, see e.g. [99, 4], we can extend the result to local nonnegative solutions. Let us consider the Minimal Dirichlet Problem, i.e. an homogeneous Dirichlet problem localized on a ball, with a smaller initial datum:

$$\begin{cases} \partial_t u = |x|^{\gamma}\nabla\cdot\left(|x|^{-\beta}\nabla u^m\right) & \text{in } Q_T=(0,T)\times B_{R_0}(x_0)\,, \\ u\,(t,x)=0 & \text{for } t>0 \text{ and } x\in\partial B_{R_0}(x_0)\,, \\ u\,(0,x)=u_0\chi_{B_R(x_0)}\geq 0 & \text{in } B_{R_0}(x_0), \text{ with } 4R<R_0. \end{cases} \tag{MDP}$$

Extinction time for MDP and minimal life time. We will show that nonnegative solutions to MDP extinguish in finite time $T=T(u_0)>0$. Moreover, T (hence its lower bound t_*) provides an estimate of the time interval for which any non-negative super-solution is strictly positive: recall that also super-solutions can extinguish in finite time. For this reason we call t_* *minimal life time* of the (super)solution u, following [24]. Estimating T in terms of the initial datum (or of the solution at a reference time) will provide estimates on the size of the intrinsic cylinders (the natural domains for positivity and Harnack estimates, whose size depends on u) for any local super-solution. Let us state the main result of this part.

Theorem 2.0.1 (Interior Lower Bounds for MDP)**.** *Let $0<4R=R_0$ and u be the solution to MDP corresponding to the initial datum $u_0\chi_{B_R(x_0)}\in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p>p_c$ when $m\in(0,m_c]$ and with $p\geq 1$ when $m\in(m_c,1)$, moreover assume that $B_{R_0}(x_0)$ satisfies either (1), (2) or (3). Then, there exist $\kappa_*,\overline{\kappa}_{p,0}>0$ depending on N,m,γ,β, given in* (2.8.5) *and* (2.2.2) *respectively, such that we have the following estimates for the extinction time $T=T(u_0)$:*

$$t_*:=\kappa_* R^{\sigma}\left(\frac{\|u_0\|_{\mathrm{L}^1_\gamma(B_R)}}{\mu_\gamma(B_R(x_0))}\right)^{1-m}\leq T\leq \frac{\mu_\gamma(B_{R_0}(x_0))^{\frac{\sigma}{(N-\gamma)}}}{\overline{\kappa}_{p,0}}\left(\frac{\|u_0\|_{\mathrm{L}^p_\gamma(B_{R_0}(x_0))}}{\mu_\gamma(B_{R_0}(x_0))^{\frac{1}{p}}}\right)^{1-m}. \tag{2.0.1}$$

Moreover, there exists $\underline{\kappa}>0$ such that

$$\inf_{x\in B_{2R}(x_0)} u(t,x)\geq \underline{\kappa}\left[\frac{\mu_\beta(B_R(x_0))}{\mu_\gamma(B_R(x_0))}\,\frac{t}{R^2}\right]^{\frac{1}{1-m}} \qquad \textit{for any } t\in[0,t_*]. \tag{2.0.2}$$

The constant $\underline{\kappa} > 0$ *depends on* N, m, γ, β *and has an (almost) explicit expression given in* (2.8.10), *and only when* $0 < m \le m_c$ *it depends also on* $\widetilde{H}_p$, *defined in* (0.0.23).

Remarks. (i) Theorem 2.0.1 holds for more general scenarios, as it will be clear by a close inspection of the proof presented in this Part. However, to simplify the presentation we have decided to state the result only under assumptions (1), (2) or (3) on page 26, since as already remarked, they represent the most relevant cases.
(ii) We recall that $\underline{\kappa}$ has a precise behaviour given in term of $\widetilde{H}_p$, and that when $m > m_c$ it does not depend on $\widetilde{H}_p$, see Remark 0.0.5 (ii) and (iii) for more details.
(iii) A closer inspection of the proofs reveals that analogous results hold for solutions of the Dirichlet problem on arbitrary bounded domains Ω and general initial data. Bounds on the extinction time similar to (2.0.1) have been obtained in [24] for the model equation, and in [52] on Riemannian manifolds.

Strategy of the proof of positivity estimates. As already explained above, it is sufficient to prove our lower estimates for solutions to the reduced problem MDP, and to avoid unnecessary technicalities, we will work with strictly positive solutions which solve a "lifted problem", cf. Subsection 2.3. The proof of our main positivity result is quite complex, as already mentioned in Subsection 0.0.2, indeed, more standard techniques seem to fail to give quantitative estimates, hence we develop a new method, that we split it into four steps:

$$\underbrace{\mathrm{L}^{-\infty} \xrightarrow[\text{Moser iteration}]{\text{Step 1, Sec. 2.4}} \mathrm{L}^{-s}_\gamma \xrightarrow[\text{Smoothing}]{\text{Step 2, Sec 2.5}} \mathrm{L}^{-\varepsilon}_\gamma \xrightarrow[\text{Parabolic John-Nirenberg}]{\text{Step 3, Sec. 2.6}}}_{\text{Prop. 2.3.1 for Lifted Problem, then let } \delta \to 0^+ \text{ for MDP (Cor. 2.7.1)}} \mathrm{L}^{\varepsilon}_\gamma \xrightarrow[\text{Smoothing}]{\text{Step 4, Sec. 2.8}} \mathrm{L}^{1}_\gamma\,.$$

The first step (Subsection 2.4) consists in proving an $\mathrm{L}^{-\infty} - \mathrm{L}^{-s}_\gamma$ estimate through a lower Moser-type iteration with negative exponents; due to the nonlinear character of our equation, such iteration does not allow one to reach all negative exponents, in contrast with what happens in the linear case here we can only reach $-s < -(1-m)$. The second step (Subsection 2.5) consists in proving quantitative $\mathrm{L}^{-s}_\gamma - \mathrm{L}^{-\varepsilon}_\gamma$ estimates for any $\varepsilon \in (0, 1-m)$. Subsection 2.6 contains the third step, a parabolic analogue of the celebrated John-Nirenberg Lemma: our Lemma provides a Reverse Hölder inequality for small exponents, in the form of an $\mathrm{L}^{-\varepsilon}_\gamma - \mathrm{L}^{\varepsilon}_\gamma$ estimate, and it holds for solutions to the (lifted) MDP; the proof relies on the monotonicity properties of the solutions to the MDP, combined with a weighted version of the John-Nirenberg Lemma, which we borrow from [80]. Corollary 2.3.1 collects all the results of the first three steps, in the form of an $\mathrm{L}^{-\infty} - \mathrm{L}^{\varepsilon}_\gamma$ smoothing effect for solutions u_δ to the δ-MDP. Next, letting $\delta \to 0$, we prove Corollary 2.7.1, which is the analogous result of Corollary 2.3.1 for solutions of the MDP. Subsection 2.8 contains the fourth and last step, namely $\mathrm{L}^{\varepsilon}_\gamma - \mathrm{L}^{1}_\gamma$ estimates, see Lemma 2.8.1; gathering all the previous results, we finally obtain the $\mathrm{L}^{-\infty} - \mathrm{L}^{1}_\gamma$ estimates in Corollary 2.8.2. The proof of Theorems 2.0.1 and 0.0.4 is contained in Section 2.9.

2.1 Basic properties of solutions to the Minimal Dirichlet Problem

We summarize here the standard properties of solutions to the MDP which will be used in what follows.

Proposition 2.1.1. *Let* $u_0 \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ *with* $p > p_c$ *when* $m \in (0, m_c]$ *and with* $p \ge 1$ *when* $m \in (m_c, 1)$. *Then there exists a unique strong solution to the problem MDP and the following properties hold:*

(i) *There exists* $\overline{\kappa}_{12} = \overline{\kappa}_{12}(\gamma, \beta, N, m, p) > 0$ *such that for any* $t > 0$

$$\|u(t)\|_{\mathrm{L}^\infty(B_{R_0}(x_0))} \le \overline{\kappa}_{12} \frac{\|u_0\|^{\sigma p \vartheta_p}_{\mathrm{L}^p_\gamma(B_R(x_0))}}{t^{(N-\gamma)\vartheta_p}}, \tag{2.1.1}$$

where $\vartheta_p = \frac{1}{\sigma p - (N-\gamma)(1-m)}$ *and* $\sigma = 2 + \beta - \gamma$.

(ii) *For all* $t > 0$ *we have* $\|u(t)\|_{\mathrm{L}^p_\gamma(B_{R_0}(x_0))} \le \|u_0\|_{\mathrm{L}^p_\gamma(B_{R_0}(x_0))}$.

(iii) *For all* $t > 0$, *the function* $t \to u(t,x) t^{-\frac{1}{1-m}}$ *is non-increasing, for almost any* $x \in B_{R_0}(x_0)$.

Remark. The constant $\overline{\kappa}_{12}$ may be chosen to be independent of $B_{R_0}(x_0)$, as explained in the proof.

Proof. Existence and uniqueness of strong solutions follow by minor modifications of standard arguments, cf. [4]. Also properties (ii) and (iii) follow by standard arguments that can be found in [4]. The upper bounds (i) hold as a consequence of the smoothing effects for the Cauchy problem on the whole space, namely inequality (2.1.1) with the norms taken on $\mathbb{R}^N$. The proof of such global estimates is easier than its local counterpart: on one hand, such bounds can be proven directly by doing a Moser iteration, and then noticing that the solution to the MDP (extended to zero outside of $B_{R_0}(x_0)$) is a subsolution to the Cauchy problem for WFDE posed on $\mathbb{R}^N$. On the other hand, such upper bounds can be deduced from the local upper estimates of Theorem 1.0.1 by letting $R_0 \to \infty$ in inequality (1.0.1) (in such limit the constant $\overline{\kappa}_8$ becomes independent of R_0 and the second term vanishes), so we obtain (2.1.1); in the latter case, assumptions (1),(2), (3) do not play an essential role, since we can always consider $x_0 = 0$. □

2.2 Proof of the bounds (2.0.1) on the extinction time for MDP.

We now prove the two sided estimate (2.0.1) on the extinction time. While the lower bounds follows from Proposition 1.2.1, the upper bounds require the following Proposition, in which we show that Sobolev and Poincaré inequalities imply extinction in finite time for solutions to the MDP, as already observed in [52, 24] for the model equation ($\beta = \gamma = 0$) both in Euclidean and Riemannian manifolds settings.

Proposition 2.2.1. *Let* u *be the solution to the MDP, corresponding to* $u_0 \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ *with* $p \ge p_c \vee 1$. *Then, for all* $q > 1$ *there exists a constant* $\overline{\kappa}_q > 0$, *such that for all* $0 \le \tau \le t$ *we have*

$$\|u(t)\|^{1-m}_{\mathrm{L}^q_\gamma(B_{R_0}(x_0))} \le \|u(\tau)\|^{1-m}_{\mathrm{L}^q_\gamma(B_{R_0}(x_0))} - \overline{\kappa}_q (t - \tau), \tag{2.2.1}$$

as long as the right-hand side is nonnegative where the constant $\overline{\kappa}_q$ *is given by*

$$\overline{\kappa}_q = \overline{\kappa}_{q,0} \mu_\gamma(B_{R_0}(x_0))^{-\frac{\sigma}{(N-\gamma)}\left(1 - \frac{p_c}{q}\right)} \qquad \text{with} \qquad \overline{\kappa}_{q,0} = \frac{4(q-1)(1-m)}{\overline{\kappa}_{13}^2 (q+m-1)^2}, \tag{2.2.2}$$

where $\overline{\kappa}_{13}$ *depends only on geometrical quantities, but not on* R_0; *we recall that* $\sigma = 2 + \beta - \gamma$.

Proof. We will write B_{R_0} instead of $B_{R_0}(x_0)$ and L^q_α instead of $\mathrm{L}^q_\alpha(B_{R_0})$ when no confusion arises. We combine weighted Sobolev and Poincaré inequalities (CKNI) via Hölder's inequality as follows:

$$\|f\|_{\mathrm{L}^s_\gamma} \le \|f\|^{1-\theta}_{\mathrm{L}^2_\gamma} \|f\|^{\theta}_{\mathrm{L}^{r^*}_\gamma} \le \overline{\kappa}_{13} \mu_\gamma(B_{R_0})^{\frac{\sigma}{2(N-\gamma)}(1-\theta)} \|\nabla f\|_{\mathrm{L}^2_\beta}, \tag{2.2.3}$$

for any $s \in (2, r^*)$ for any function $f \in \mathcal{D}_{\gamma,\beta}$. Letting now $s = \frac{2q}{q+m-1}$, $f = u^{\frac{q+m-1}{2}}$ and $\theta = p_c/q$, and recalling that $s < r^*$ if and only if $q > p_c$, we have that inequality (2.2.3) implies

$$\|f\|^2_{\mathrm{L}^s_\gamma} = \|u\|^{q\left[1-\frac{1-m}{q}\right]}_{\mathrm{L}^q_\gamma} \le \overline{\kappa}^2_{13}\mu_\gamma(B_{R_0})^{\frac{\sigma}{(N-\gamma)}(1-\frac{p_c}{q})}\|\nabla u^{\frac{q+m-1}{2}}\|^2_{\mathrm{L}^2_\beta}\,.$$

Next, we formally take the derivative of the L^q_γ norm of u to get the following differential inequality

$$\begin{aligned}\frac{d}{dt}\|u\|^q_{\mathrm{L}^q_\gamma} &= -\frac{4q(q-1)}{(q+m-1)^2}\|\nabla u^{\frac{q+m-1}{2}}\|^2_{\mathrm{L}^2_\beta} \\ &\le -\frac{4q(q-1)}{(q+m-1)^2}\overline{\kappa}^2_{13}\mu_\gamma(B_{R_0})^{-\frac{\sigma}{(N-\gamma)}(1-\theta)}\|u\|^{q\left(1-\frac{1-m}{q}\right)}_{\mathrm{L}^q_\gamma}\,.\end{aligned}$$

Integrating this last inequality in $[\tau, t] \subset [0, T]$ we get (2.2.1). A rigorous proof (long and technical, but nowadays standard) can be done by using energy inequalities and Grownwall-type arguments. □

Proof of Inequalities 2.0.1. As an immediate corollary of inequalities (2.2.1), there exists the extinction time $T > 0$; moreover, it satisfies the upper bound 2.0.1, which is nothing but inequality (2.2.1) with $\tau = 0$ and $t = T$. The lower bound follows by letting $t = 0$ and $\tau = T$ in formula (1.2.2). □

2.3 Lifted problem and a first positivity result

In this section we address the question of proving quantitative estimates of positivity for the MDP. We begin by introducing the following "lifted" problem: let $\delta > 0$, $0 < R \le R_0$

$$\begin{cases}\partial_t u_\delta = |x|^\gamma \nabla \cdot \left(|x|^{-\beta}\nabla u^m_\delta\right) & \text{in } (0,\infty)\times B_{R_0}(x_0)\,,\\ u_\delta = \delta & \text{on } (0,\infty)\times \partial B_{R_0}(x_0)\,,\\ u_\delta(0,x) = u_0(x)\chi_{B_R(x_0)}(x) + \delta & \text{for } x \in B_{R_0}(x_0).\end{cases} \tag{δ-MDP}$$

The results of Proposition 2.1.1 hold also for solutions to the δ-MDP, more precisely, (ii) and (iii) hold in the same form, while (i) holds with an extra δ factor on the right-hand side, as explained below; the proofs of the latter facts are a straightforward modification of the case $\delta = 0$, indeed, it is clear that $v_\delta = u_\delta - \delta$, solves an homogeneous Dirichlet problem with a regularized nonlinearity: $\partial_t v_\delta = |x|^\gamma \nabla \cdot \left(|x|^{-\beta}\nabla(v^m_\delta + \delta^m) - \delta^m\right)$, as in Appendix B of [86], or as in Chapter 5 of [4], where even more general nonlinearities are treated. However, we recall explicitly the two main estimates that we will use in what follows: the time monotonicity property (iii) of Proposition 2.1.1, namely the fact that $t \to u_\delta(\cdot,t)t^{-\frac{1}{1-m}}$ is non-increasing in t; the upper bounds, namely, there exists $\overline{\kappa}_{12}$, given in (2.1.1), such that for any $(t,x) \in (0,\infty)\times B_{R_0}(x_0)$

$$\delta \le u_\delta(t,x) \le \frac{2\overline{\kappa}_{12}}{t^{(N-\gamma)\vartheta_p}}\|u_0\chi_{B_R(x_0)} + \delta\|^{p\sigma\vartheta_p}_{\mathrm{L}^p_\gamma(B_{R_0}(x_0))} + 2\delta := M_p(u_0,\delta,t). \tag{2.3.1}$$

The next Proposition proves $\mathrm{L}^{-\infty} - \mathrm{L}^\varepsilon$ estimates for solutions to δ-MDP; this is the core of the proof of the main Theorem 2.0.1 and it contains the first three steps explained above. In what follows, we need to assume a certain weighted L^p integrability on the initial datum, otherwise the above inequality may fail, as thoroughly explained before. It is convenient, even if not strictly needed, to assume u_δ to be bounded: this will simplify the proofs of the lower iteration.

Proposition 2.3.1 ($\mathrm{L}^{-\infty} - \mathrm{L}^{\varepsilon}$ estimates for δ-MDP). *Let u_δ be a strong (super)solution to δ-MDP on $(0,\infty)\times B_{R_0}(x_0)$, corresponding to $u_0\chi_{B_R(x_0)} \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p > p_c$ when $m \in (0, m_c]$ and with $p \ge 1$ when $m \in (m_c, 1)$. For any $0 \le T_0 < T_1 < T_2 < T_3$, such that $T_3 - T_2 \ge T_1 - T_0$, for any $x_0 \in \mathbb{R}^N$ and for any $0 < R_2 < R_1 < R$ such that $0 \notin \overline{B_{R_1}(x_0)\setminus B_{R_2}(x_0)}$, and for any $\varepsilon > 0$ such that*

$$0 < \varepsilon < \nu_\delta \wedge (1-m) \qquad \text{with}$$

$$\nu_\delta = \frac{1}{\overline{\kappa}_{14}\,\overline{\kappa}_6}\left[1 + \frac{R_1^\sigma}{T_0}\left(\frac{|x_0|}{R_1}\vee 1\right)^{\beta-\gamma} M_p(u_0,\delta,T_0)^{1-m}\right]^{-1/2}, \tag{2.3.2}$$

there exist $s_\varepsilon > 1-m$ and $\underline{\kappa}_2 > 0$ such that

$$\begin{aligned}\inf_{(T_1,T_2]\times B_{R_2}(x_0)} u_\delta &\ge \underline{\kappa}_2\left[\left(1 + M_p(u_0,\delta,T_0)^{1-m}\right)\right]^{\eta_\varepsilon + \frac{s_\varepsilon}{s_\varepsilon+m-1}\zeta_\varepsilon}(T_2-T_0)^{-\frac{1}{s_\varepsilon+m-1}}\\
&\times\left[\left(\mu_\gamma(B_{R_2}(x_0))^{-\frac{\sigma}{N-\gamma}} \vee \left(\frac{h_\sigma(R_1,R_2,x_0)}{(R_1-R_2)^\sigma} + \frac{1}{T_1-T_0}\right)\right)\right]^{\eta_\varepsilon + \frac{s_\varepsilon}{s_\varepsilon+m-1}\zeta_\varepsilon}\\
&\times\left[\left(\frac{T_0}{T_3}\right)^{\frac{1}{1-m}}\mu_\gamma(B_{R_1}(x_0))^{-\frac{2}{s_\varepsilon}}\left(\int_{B_{R_1}(x_0)} u_\delta(T_3,x)^\varepsilon|x|^{-\gamma}\,\mathrm{d}x\right)^{\frac{1}{\varepsilon}}\right]^{\frac{s_\varepsilon}{s_\varepsilon+m-1}}\end{aligned} \tag{2.3.3}$$

where k_ε is the smallest $k \in \mathbb{N}$ such that $(r^*/2)^k\varepsilon > 1-m$ and

$$s_\varepsilon := \left(\frac{r^*}{2}\right)^{k_\varepsilon}\varepsilon\,, \quad \eta_\varepsilon := -\frac{1}{s_\varepsilon+m-1}\left(\frac{N-\gamma}{2+\beta-\gamma}+1\right) < 0\,, \quad \zeta_\varepsilon := -\frac{1}{\varepsilon}\frac{1-\left(\frac{2}{r^*}\right)^{k_\varepsilon}}{1-\frac{2}{r^*}} < 0\,, \tag{2.3.4}$$

with h_σ, M_p and r^* defined in (1.1.1), (2.3.1) *and* (CKNI) *respectively; we also provide the expression of the constant $\underline{\kappa}_2 = \underline{\kappa}_3\underline{\kappa}_4^{\frac{s_\varepsilon}{s_\varepsilon+m-1}}\overline{\kappa}_7^{-\frac{2}{s_\varepsilon+m-1}}2^{(2\vee\sigma)(\eta_\varepsilon+\frac{s_\varepsilon}{s_\varepsilon+m-1}\zeta_\varepsilon)}$ with $\underline{\kappa}_3, \underline{\kappa}_4, \overline{\kappa}_7$ depending only on $N, m, p, \beta, \gamma, \varepsilon$ and given in* (2.4.1), (2.5.2), (0.0.31) *respectively; $\overline{\kappa}_6$ given in Lemma 0.0.13 depends on N, γ, $\overline{\kappa}_{14}$ given in Lemma 2.6.2 depends on N, γ, β, m.*

The next four subsections will be devoted to the proof of the above Proposition.

2.4 Step 1. Lower Moser Iteration.

In this section we prove $\mathrm{L}^{-\infty} - \mathrm{L}^{-s}$ smoothing effects by means of a lower Moser-type iteration. We will cover more general cases than (1), (2) or (3). We are going to prove a priori estimates of the solution to the δ-MDP, and such bounds involve the quantity h_σ that under our assumptions will always be bounded.

Proposition 2.4.1 (Nonlinear case). *Let u be a strong (super)solution to δ-MDP on $(T_0, T_2)\times B_{R_0}(x_0)$, corresponding to $u_0 \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p > p_c$ when $m \in (0, m_c]$ and with $p \ge 1$ when $m \in (m_c, 1)$. Then, for any $s > 1-m$, for any $0 < R_2 < R_1 \le R_0$ such that $0 \notin \overline{B_{R_1}(x_0)\setminus B_{R_2}(x_0)}$*

and for any $T_1 \in (T_0, T_2)$, *there exists* $\underline{\kappa}_3 > 0$ *such that*

$$\inf_{(T_1,T_2]\times B_{R_2}(x_0)} u \geq \underline{\kappa}_3 \left[\left(1 + M_p(u_0,\delta,T_0)^{1-m}\right)\left(\frac{h_\sigma\left(R_1,R_2,x_0\right)}{(R_1-R_2)^\sigma} + \frac{1}{T_1-T_0}\right)\right]^{\eta_s}$$
$$\times \left(\frac{T_0}{T_2}\right)^{\frac{s}{(1-m)(s+m-1)}} (T_2-T_0)^{-\frac{1}{s+m-1}} \tag{2.4.1}$$
$$\times \left(\int_{B_{R_1}(x_0)} u^{-s}(T_2,x)|x|^{-\gamma}\,\mathrm{d}x\right)^{-\frac{1}{s+m-1}}$$

where $\eta_s := -\frac{1}{s+m-1}\left(\frac{N-\gamma}{2+\beta-\gamma}+1\right)$, h_σ *and* M_p *are defined in* (1.1.1) *and* (2.3.1) *respectively, and* $\underline{\kappa}_3 > 0$ *depends on* $s, p, N, \gamma, \beta, m$.

Remarks. (i) We recall that the technical assumption $0 \notin \overline{B_{R_1}(x_0) \setminus B_{R_2}(x_0)}$ is needed only to guarantee that the quantity h_σ is finite.
(ii) The above estimate also holds when $m = 1$, in which case we can take any $s > 0$, see Proposition 3.1.5. This fact considerably simplifies the proof of lower bounds in the linear case: the lower bound is formulated in terms of a space-time integral on a bigger parabolic cylinder, and this is the classical result in linear parabolic equations, see for instance [62, 63, 64, 65, 72, 71, 73, 74]. We refer to Section 3.1 for further details.

Proof of Proposition 2.4.1. Throughout the proof u_δ will be a strong (super)solution to δ-MDP on a generic cylinder $Q := (T_0, T) \times B_R(x_0)$ where T and R are fixed, and will be chosen at the end of the proof; we will write $u = u_\delta$, and $B_R = B_R(x_0)$ ($B_{R_1} = B_{R_1}(x_0)$ resp.) when no confusion arises. We recall that $\sigma = 2 + \beta - \gamma > 0$, $r^* = 2(N-\gamma)/(N-2-\beta)$, and we set $q = (N-\gamma)/\sigma$. We will split the proof into several steps.

• Step 1. *Preparation of the iteration step.* Let $0 < R_1 < R_0$ and $0 \leq T_0 < T_1 < T$ and define $Q_0 := Q$ and $Q_1 := (T_1, T] \times B_{R_1}(x_0)$. We are going to prove the following inequality, with $a \in (1, r^*/2)$:

$$\iint_{Q_1} u^{a(m-p-1)}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t \leq [2S_{\gamma,\beta}]^2\,[2c(m,p)]^{1+\frac{1}{q}}\left[\frac{h_\sigma\left(R_0,R_1,x_0\right)}{(R_0-R_1)^\sigma} + \frac{1}{T_1-T_0}\right]^{1+\frac{1}{q}}$$
$$\times \left[1 + M_{\widetilde{p}}(u_0,\delta,T_0)^{1-m}\right]^{1+\frac{1}{q}}\left[\iint_{Q_0} u^{-p+m-1}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{1+\frac{1}{q}}. \tag{2.4.2}$$

To prove the above inequality, we first recall the CKNI inequality (1.3.2) with $f^2 = u^{m-p-1}$ and $p > 0$:

$$\iint_{Q_1} u^{a(m-p-1)}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}$$
$$\leq 2S_{\gamma,\beta}^2\left[\iint_{Q_1}\left(u^{m-p-1}|x|^{-\gamma} + \mu_\gamma(B_{R_1})^{\frac{\sigma}{N-\gamma}}\left|\nabla u^{\frac{m-p-1}{2}}\right|^2\ |x|^{-\beta}\right)\mathrm{d}x\,\mathrm{d}t\right] \tag{2.4.3}$$
$$\times\, \mu_\gamma(B_{R_1})^{-\frac{1}{q}}\left[\sup_{t\in(T_1,T]}\int_{B_{R_1}} u^{(m-p-1)(a-1)q}|x|^{-\gamma}\,\mathrm{d}x\right]^{\frac{1}{q}}.$$

We are going to estimate the two terms on the right-hand side of the above inequality, by means of a modified form of the lower energy estimates (1.1.4), which easily follows by using (2.3.1): for any

$p > 0$ and any $T_1 \in (T_0, T)$ and $R_1 \in (0, R)$ we have

$$\begin{aligned}
&\sup_{\tau\in[T_1,T]} \int_{B_{R_1}(x_0)} u(\tau,x)^{-p}\frac{\mathrm{d}x}{|x|^\gamma} + \int_{T_1}^{T}\int_{B_{R_1}(x_0)} \left|\nabla u^{\frac{-p+m-1}{2}}\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta} \\
&\le c_3\left[1 + M_{\widetilde{p}}(u_0,\delta,T_0)^{1-m}\right] \\
&\quad\times \left[\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma} + \frac{1}{T_1-T_0}\right] \int_{T_0}^{T}\int_{B_R(x_0)} u^{-p+m-1}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma},
\end{aligned} \tag{2.4.4}$$

where $M_{\widetilde{p}}(u_0,\delta,T_0)$ is defined in (2.3.1), and $\widetilde{p} > p_c$ when $m \in (0, m_c]$ and with $\widetilde{p} \ge 1$ when $m \in (m_c, 1)$; h_σ is defined in (1.1.1), and $c_3 > 0$ depends on m, p, N, and is given in the energy inequality (1.1.4). We estimate now the first term of (2.4.3) using (2.4.4):

$$\iint_{Q_1} \left(u^{m-p-1}|x|^{-\gamma} + \mu_\gamma(B_{R_1})^{\frac{\sigma}{N-\gamma}}\left|\nabla u^{\frac{m-p-1}{2}}\right|^2 |x|^{-\beta}\right)\mathrm{d}x\,\mathrm{d}t \le J\iint_{Q_0} u^{m-p-1}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\,, \tag{2.4.5}$$

where

$$J := 2c_3\mu_\gamma(B_{R_1})^{\frac{\sigma}{N-\gamma}}\left[1 + M_{\widetilde{p}}(u_0,\delta,T_0)^{1-m}\right]\left[\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma} + \frac{1}{T_1-T_0}\right] \ge 1.$$

Note that it is not restrictive to assume that $J \ge 1$, by an argument similar to the footnote related to formula (1.3.6).

We estimate the second term in the right-hand side of (2.4.3) using again (2.4.4), observing that we can always choose $a \in (1, r^*/2)$ such that $(m-p-1)(a-1)q = -p$ to get

$$\begin{aligned}
\left[\sup_{t\in(T_1,T]}\int_{B_{R_1}} u^{-p}|x|^{-\gamma}\,\mathrm{d}x\right]^{\frac{1}{q}} &\le \left[c_3\left(1 + M_{\widetilde{p}}(u_0,\delta,T_0)^{1-m}\right)\right]^{\frac{1}{q}} \\
&\quad\times\left[\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma} + \frac{1}{T_1-T_0}\right]^{\frac{1}{q}} \\
&\quad\times\left[\iint_{Q_0} u^{-p+m-1}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.4.6}$$

Combining inequalities (2.4.3), (2.4.5) and (2.4.6) we get (2.4.2).

• Step 2. *The k^{th} iteration step.* We first define an increasing sequence of exponents p_k, recalling that $q = r^*/(r^*-2)$ and $p_0 > 0$, set

$$p_{k+1} := \left(1 + \frac{1}{q}\right)p_k = \left(1+\frac{1}{q}\right)^{k+1} p_0 \xrightarrow[k\to+\infty]{} +\infty\,.$$

Next, we define the cylinders $Q_k := (T_k, T] \times B_{R_k}(x_0)$ as in (1.3.9), so that $Q \supset Q_k \supset Q_{k+1} \to Q_\infty$; we have chosen a decreasing sequence of radii $R_0 > R_k > R_{k+1} \to R_\infty$ and an increasing sequence of times $T_0 < T_k < T_{k+1} \to T_\infty$ as in (1.3.10), namely $R_k - R_{k+1} = C_1(R_0 - R_\infty)(k+1)^{-\alpha}$ and $T_{k+1} - T_k = C_2(T_\infty - T_0)(k+1)^{-\alpha\sigma}$, where $\alpha = 2 \vee \sigma^{-1}$ and $C_1, C_2 > 0$ are as in (1.3.11).

Plugging all the above defined quantities in inequality (2.4.2), the k^{th} iteration step reads

$$
\begin{aligned}
&\left[\iint_{Q_{k+1}} u^{-(p_{k+1}+1-m)}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{-\frac{1}{p_{k+1}+1-m}} \\
&\ge [2S_{\gamma,\beta}]^{-\frac{2}{p_{k+1}+1-m}}\left[2c_3\left(1+M_{\widetilde{p}}(u_0,\delta,T_k)^{1-m}\right)\right]^{-\frac{1}{p_{k+1}+1-m}\left(1+\frac{1}{q}\right)} \\
&\quad\times\left[\frac{h_\sigma(R_k,R_{k+1},x_0)}{(R_k-R_{k+1})^\sigma}+\frac{1}{T_{k+1}-T_k}\right]^{-\frac{1}{p_{k+1}+1-m}\left(1+\frac{1}{q}\right)} \\
&\quad\times\left[\iint_{Q_k} u^{-(p_k+1-m)}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right]^{-\frac{1}{p_{k+1}+1-m}\left(1+\frac{1}{q}\right)}.
\end{aligned}
\tag{2.4.7}
$$

Bounds for the constants. It is convenient to bound the constants appearing in (2.4.7) by a quantity which does not depend on p, but only on m,γ,β,R_∞ and R_0. Recall that $c_3 := 2K_\psi\ c_{m,p_k}$ is given in the energy inequality (1.1.4): while $K_\psi > 0$ depends only on N, the quantity $c_{m,p_k} = \frac{p_k+1}{p_k}\left(\frac{p_k+1}{p_k}\wedge\frac{2m(p_k+1)^2}{(m-p_k-1)^2}\right)^{-1}$ needs to be bounded uniformly for all $k\ge 0$; since $p_k > p_0 > 0$ it is easy to show that we have $c_{m,p_k}\le(1+p_0^{-1})/(1\wedge 4m)$, so that $2c_3 = 2c_3(p_k)\le\underline{c}=\underline{c}(m,N,p_0)<+\infty$; hence $2c_3(p_k)\le\underline{c}$ can be bounded uniformly by a constant that depends only on N,m,p_0. As in (1.3.13), also $h_\sigma(R_k,R_{k+1},x_0)$ can be bounded by a quantity depending only on σ,R_0 and R_∞, as follows:

$$
h_\sigma\left(R_k,R_{k+1},x_0\right)\le h_\sigma\left(R_0,R_\infty,x_0\right)(k+1)^{\alpha(2-\sigma)_+}\,C_1^{-(2-\sigma)_+},
$$

where $C_1>0$ is as in (1.3.11). Moreover, by (2.3.1) we have that $M_{\widetilde{p}}(u_0,\delta,T_k)\le M_{\widetilde{p}}(u_0,\delta,T_0)$, since $T_0<T_k$. Finally, we can rewrite the k^{th} iteration step (2.4.7) as follows:

$$
\left[\iint_{Q_{k+1}} u^{-(p_{k+1}+1-m)}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right]^{\frac{-1}{p_{k+1}+1-m}}\ge I_{k+1}^{\frac{-1}{p_{k+1}+1-m}}\left[\iint_{Q_k} u^{-(p_k+1-m)}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right]^{\frac{-\left(1+\frac{1}{q}\right)}{p_{k+1}+1-m}},
\tag{2.4.8}
$$

where the constant I_{k+1} is bounded by

$$
\begin{aligned}
I_{k+1}&\le[2S_{\gamma,\beta}]^2\left[\underline{c}\left(\frac{h_\sigma\left(R_0,R_\infty,x_0\right)C_1^{-\frac{b}{\alpha}}}{(R_0-R_\infty)^\sigma}+\frac{C_2^{-1}}{T_\infty-T_0}\right)\left(1+M_{\widetilde{p}}(u_0,\delta,T_0)^{1-m}\right)\right]^{1+\frac{1}{q}}(k+1)^{b\left(1+\frac{1}{q}\right)} \\
&\le[2S_{\gamma,\beta}]^2\left[\underline{c}(C_1^{-\frac{b}{\alpha}}\vee C_2^{-1})\left(\frac{h_\sigma\left(R_0,R_\infty,x_0\right)}{(R_0-R_\infty)^\sigma}+\frac{1}{T_\infty-T_0}\right)\left(1+M_{\widetilde{p}}(u_0,\delta,T_0)^{1-m}\right)\right]^{1+\frac{1}{q}}(k+1)^{b\left(1+\frac{1}{q}\right)} \\
&:= J_0J_1^{1+\frac{1}{q}}(k+1)^{b\left(1+\frac{1}{q}\right)},
\end{aligned}
\tag{2.4.9}
$$

where $b=2\alpha$ if $0<\sigma<2$, $b=\sigma\alpha$ otherwise. The assumption (1.3.11) on α ensures that $b>1$.

• STEP 3. *The iteration.* We now iterate inequalities (2.4.8) to get

$$
\begin{aligned}
&\left[\iint_{Q_{k+1}} u^{-(p_{k+1}+1-m)}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right]^{\frac{-1}{p_{k+1}+1-m}} \\
&\ge\prod_{j=1}^{k+1} I_j^{\frac{-\left(1+\frac{1}{q}\right)^{k+1-j}}{p_{k+1}+1-m}}\left[\iint_{Q_0} u^{-(p_0+1-m)}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right]^{\frac{-\left(1+\frac{1}{q}\right)^{k+1}}{p_{k+1}+1-m}}.
\end{aligned}
\tag{2.4.10}
$$

We estimate the product appearing in the above inequality as follows:

$$\begin{aligned}\prod_{j=1}^{k+1} I_j^{\frac{-1}{p_{k+1}+1-m}\left(1+\frac{1}{q}\right)^{k+1-j}} &\ge \left[J_0 J_1^{1+\frac{1}{q}}\right]^{\frac{-1}{p_{k+1}+1-m}\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^j} \left[(k+1)^{b'}\right]^{\frac{-1}{p_{k+1}+1-m}} \cdots \left[2^{b'}\right]^{\frac{-1}{p_{k+1}+1-m}\left(1+\frac{1}{q}\right)^{k-1}} \\ &= \left[J_0 J_1^{1+\frac{1}{q}}\right]^{\frac{-1}{p_{k+1}+1-m}\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^j} \prod_{j=1}^{k+1} j^{\frac{-b'}{p_{k+1}+1-m}\left(1+\frac{1}{q}\right)^{k+1-j}},\end{aligned}$$

where $b' = b\left(1+\frac{1}{q}\right)$. Recalling that $p_k = (1+\frac{1}{q})^k p_0$ we get

$$\frac{-1}{p_{k+1}+1-m}\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^j \xrightarrow[k\to+\infty]{} -\frac{q}{p_0} \qquad \text{and} \qquad \frac{p_{k+1}+1-m}{\left(1+\frac{1}{q}\right)^{k+1}} \xrightarrow[k\to+\infty]{} p_0\,.$$

Moreover, it is easy to show that

$$\begin{aligned}\lim_{k\to+\infty}\prod_{j=1}^{k+1} j^{\frac{b'}{p_0(1+\frac{1}{q})^{k+1}+1-m}\left(1+\frac{1}{q}\right)^{k+1-j}} &\le \lim_{k\to+\infty}\exp\left(\frac{b'}{p_0+\frac{(1-m)}{(1+\frac{1}{q})^{k+1}}}\sum_{j=1}^{\infty}\left(\frac{q}{q+1}\right)^j \log j\right) \\ &\le (c'')^{\frac{q}{p_0}} < +\infty\,.\end{aligned}$$

Taking the limit in (2.4.10) as $k\to\infty$ we obtain

$$\inf_{Q_\infty} u \ge \left[c''\, J_0 J_1^{1+\frac{1}{q}}\right]^{-\frac{q}{p_0}} \left(\iint_{Q_0} u^{-(p_0+1-m)}|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right)^{-\frac{1}{p_0}}. \tag{2.4.11}$$

Note that J_0 and c'' depend only on m, p_0, N, γ and β, while J_1 depends also on $R_0 - R_\infty$ and $T_\infty - T_0$. • STEP 4. The goal of this step is to show that estimate (2.4.11) implies estimate (2.4.1). To this end, we use the time monotonicity properties of the solution, namely the fact that $t \to u(t,x)t^{-\frac{1}{1-m}}$ is non-increasing in time for almost every $x \in B_{R_0}(x_0)$. Recalling that $p_0 > 0$, we get for any $t \in [T_0, T]$

$$\int_{B_{R_0}(x_0)} u(t,x)^{m-1-p_0}|x|^{-\gamma}\,\mathrm{d}x \le \left(\frac{T}{T_0}\right)^{\frac{p_0+1-m}{1-m}} \int_{B_{R_0}(x_0)} u(T,x)^{m-p_0-1}|x|^{-\gamma}\,\mathrm{d}x\,.$$

Hence, (2.4.11) can be estimated as follows (recall that J_0, J_1 are defined in (2.4.9))

$$\inf_{Q_\infty} u \ge \underline{\kappa}\left[J_0 J_1^{1+\frac{1}{q}}\right]^{-\frac{q}{p_0}} \left(\frac{T_0}{T}\right)^{\frac{p_0+1-m}{p_0(1-m)}} (T-T_0)^{-\frac{1}{p_0}} \left(\int_{B_{R_0}(x_0)} u^{m-p_0-1}(T,x)|x|^{-\gamma}\,\mathrm{d}x\right)^{-\frac{1}{p_0}}.$$

This is exactly (2.4.1), with $R_0 \rightsquigarrow R_1$, $R_\infty \rightsquigarrow R_2$, $T_\infty \rightsquigarrow T_1$, $T \rightsquigarrow T_2$ and $s := p_0+1-m$, $p := \widetilde{p}$. $\square$

Remark. Note that the estimate degenerates in the limits $m \to 1^-$ or $m \to 0^+$, indeed the term $(T_0/T)^{\frac{p_0+1-m}{p_0(1-m)}} \xrightarrow{m\to1^-} 0$; also, by (2.4.9) and previous discussions we have $J_1 \sim 1/m$, so that $J_1^{-\frac{q+1}{p_0}} \xrightarrow{m\to0^+} 0$.

2.5 Step 2. Smoothing effects for negative norms.

In the previous step we have proved an estimate of type $\mathrm{L}^{-\infty}-\mathrm{L}^{-s}$, which holds for any $s>1-m$, but this is not sufficient in order to use the reverse Hölder inequalities of Corollary 0.0.14, which may hold only for exponents close to 0. In this step we solve this issue by proving $\mathrm{L}^{-s}-\mathrm{L}^{-\varepsilon}$ estimates, for any $\varepsilon\in(0,1-m)$, through a finite iteration.

Proposition 2.5.1 ($\mathrm{L}^{-s}-\mathrm{L}^{-\varepsilon}$ Smoothing Effects). *Let u_δ be a strong (super)solution to δ-MDP on $(0,\infty)\times B_{R_0}(x_0)$, corresponding to $u_0\chi_{B_R(x_0)}\in\mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p>p_c$ when $m\in(0,m_c]$ and with $p\ge 1$ when $m\in(m_c,1)$. Let $0<R_2<R_1\le R$ be such that $0\notin\overline{B_{R_1}(x_0)\setminus B_{R_2}(x_0)}$ and let $0<T_0<T_1$. For any $\varepsilon\in(0,1-m)$ there exists $s_\varepsilon\in\left(1-m,\frac{r^*}{2}(1-m)\right]$ and $\underline{\kappa}_4>0$ such that*

$$\left[\int_{B_{R_2}(x_0)}u_\delta(T_0,x)^{-s_\varepsilon}|x|^{-\gamma}\,\mathrm{d}x\right]^{-\frac{1}{s_\varepsilon}}\ge\underline{\kappa}_4\left(\frac{T_0}{T_1}\right)^{\frac{1}{1-m}}\left[\int_{B_{R_1}(x_0)}u_\delta(T_1,x)^{-\varepsilon}|x|^{-\gamma}\,\mathrm{d}x\right]^{-\frac{1}{\varepsilon}}$$
$$\times\left[\left(1+M_p(u_0,\delta,T_0)^{1-m}\right)\left(\mu_\gamma(B_{R_2}(x_0))^{-\frac{\sigma}{N-\gamma}}\vee\left(\frac{h_\sigma(R_1,R_2,x_0)}{(R_1-R_2)^\sigma}+\frac{1}{T_1-T_0}\right)\right)\right]^{\zeta_\varepsilon},\qquad(2.5.1)$$

where k_ε is the smallest $k\in\mathbb{N}$ such that $(r^*/2)^k\varepsilon>1-m$ and

$$s_\varepsilon:=(r^*/2)^{k_\varepsilon}\varepsilon>1-m\qquad\text{and}\qquad\zeta_\varepsilon:=-\frac{1}{\varepsilon}\frac{1-\left(\frac{2}{r^*}\right)^{k_\varepsilon}}{1-\frac{2}{r^*}}$$
$$\underline{\kappa}_4:=\underline{\kappa}_4'\begin{cases}1 & \text{if } \varepsilon\neq(1-m)(2/r^*)^k \text{ for all } k\in\mathbb{N}\\ \mu_\gamma(B_{R_1})^{-\frac{1}{1-m}\left(\frac{r^*}{2}\right)^k\frac{r^*-2}{r^*+2}} & \text{if } \varepsilon=(1-m)(2/r^*)^k \text{ for some } k\in\mathbb{N},\end{cases}\qquad(2.5.2)$$

where h_σ, M_p and r^* are defined in (1.1.1), (2.3.1) *and* (CKNI) *respectively, while $\underline{\kappa}_4'$ only depends on $N,m,\beta,\gamma,\varepsilon$.*

Proof. The proof relies on a finite iteration and it is split into two steps. Let us fix $x_0\in\mathbb{R}^N$, and simply denote $B_R=B_R(x_0)$ and $u=u_\delta\ge\delta>0$, when there is no ambiguity.

• STEP 1. *Preparation of the iteration step.* Let us define $\overline{c}_1$ as

$$\overline{c}_1:=\left[1+M_p(u_0,\delta,t_0)^{1-m}\right]\left[\frac{h_\sigma(R_1,R_2,x_0)}{(R_1-R_2)^\sigma}+\frac{1}{t_1-t_0}\right].\qquad(2.5.3)$$

We are going to prove that for any $s\in(0,1-m)$, for any $0<R_2<R_1\le R\le R_0$ and $0<t_0<t_1$, there exists a constants $\underline{\kappa}_4>0$, depending on q,γ,β,N,m,x_0 and R_1,R_2,t_0,t_1 and the norm of the initial data $\|u_0\|_{\mathrm{L}^p_\gamma(B_R(x_0))}$ such that

$$\left[\int_{B_{R_2}(x_0)}u(t_0,x)^{\frac{-r^*s}{2}}|x|^{-\gamma}\,\mathrm{d}x\right]^{-\frac{2}{r^*s}}\ge\underline{c}\left[\int_{B_{R_1}(x_0)}u(t_1,x)^{-s}|x|^{-\gamma}\,\mathrm{d}x\right]^{-\frac{1}{s}},\qquad(2.5.4)$$

where $\underline{c}=\left[\left(\mu_\gamma(B_{R_2})^{\frac{-\sigma}{(N-\gamma)}}\vee\overline{c}_1\right)4S^2_{\gamma,\beta}\right]^{-1/s}(t_1/t_0)^{1/(1-m)}$, $\overline{c}_1$ is as in (2.5.3) and $S_{\gamma,\beta}$ is as in (CKNI2).

Note that while $s\in(0,1-m)$, in general $r^*s/2$ may be bigger than $1-m$: we will exploit this fact in the next step. We first integrate in time inequality (CKNI2) applied to $f = u^{\frac{\tilde{p}+m-1}{2}}$, with $\tilde{p} \in (0, 1-m)$, to get

$$\begin{aligned}&\int\limits_{t_0}^{t}\left(\int\limits_{B_{R_2}} u^{\frac{r^*}{2}(\tilde{p}+m-1)}\frac{\mathrm{d}x}{|x|^\gamma}\right)^{\frac{2}{r^*}}\mathrm{d}\tau\\ &\le 2\overline{S}^2_{\gamma,\beta}\left[\mu_\gamma(B_{R_2})^{\frac{-\sigma}{(N-\gamma)}}\int\limits_{t_0}^{t}\int\limits_{B_{R_2}} u^{\tilde{p}+m-1}\frac{\mathrm{d}x\,\mathrm{d}\tau}{|x|^\gamma}+\int\limits_{t_0}^{t}\int\limits_{B_{R_2}}\left|\nabla u^{\frac{\tilde{p}+m-1}{2}}\right|^2\frac{\mathrm{d}x\,\mathrm{d}\tau}{|x|^\beta}\right].\end{aligned} \tag{2.5.5}$$

Recalling that by (2.3.1) we have $u(t,x) \le M_p(u_0,\delta,t_0)$, for all $t \ge t_0$ and $x \in B_R(x_0)$, then the energy inequality (1.1.3) implies for all $\tilde{p} \in (0, 1-m)$ (recall that $\tilde{p}+m-1 < 0$ and $u \ge \delta > 0$)

$$\int_{t_0}^{t}\int_{B_{R_2}}\left|\nabla u^{\frac{\tilde{p}+m-1}{2}}\right|^2\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta} \le \overline{c}\int_{t_0}^{t_1}\int_{B_{R_1}} u^{\tilde{p}+m-1}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}.$$

where $t \in (t_0, t_1)$ and

$$\overline{c} = c_2\left[1 + M_p(u_0,\delta,t_0)^{1-m}\right]\left[\frac{h_\sigma(R_1,R_2,x_0)}{(R_1-R_2)^\sigma} + \frac{1}{t_1-t}\right],$$

h_σ and M_p are defined in (1.1.1) and (2.3.1) and $c_2 = c_2(m,\tilde{p}) > 0$ is as in (1.1.3). Combining the two above inequalities we obtain

$$\int_{t_0}^{t}\left(\int_{B_{R_2}} u^{\frac{r^*(\tilde{p}+m-1)}{2}}\frac{\mathrm{d}x}{|x|^\gamma}\right)^{\frac{2}{r^*}}\mathrm{d}\tau \le 2\overline{S}^2_{\gamma,\beta}\left(\mu_\gamma(B_{R_2})^{\frac{-\sigma}{(N-\gamma)}} \vee \overline{c}\right)\int_{t_0}^{t_1}\int_{B_{R_1}(x_0)} u^{\tilde{p}+m-1}\frac{\mathrm{d}x\,\mathrm{d}\tau}{|x|^\gamma}. \tag{2.5.6}$$

We recall that $t \to u(t,x)t^{-\frac{1}{1-m}}$ is non-increasing in time for almost every $x \in B_{R_0}(x_0)$, hence we can estimate the two sides of the above inequality: the left-hand side can be estimated from below

$$\begin{aligned}&\int_{t_0}^{t}\left[\int_{B_{R_2}(x_0)} u(\tau,x)^{\frac{r^*(\tilde{p}+m-1)}{2}}\frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{2}{r^*}}\mathrm{d}\tau\\ &\ge \tilde{c}\,\frac{t^{\frac{\tilde{p}}{1-m}} - t_0^{\frac{\tilde{p}}{1-m}}}{t_0^{\frac{\tilde{p}+m-1}{1-m}}}\left[\int_{B_{R_2}(x_0)} u(t_0,x)^{\frac{r^*(\tilde{p}+m-1)}{2}}\frac{\mathrm{d}x}{|x|^\gamma}\right]^{\frac{2}{r^*}},\end{aligned} \tag{2.5.7}$$

where $\tilde{c} = (1-m)/\tilde{p}$. Analogously, we can estimate the right-hand side of (2.5.6) from above,

$$\int_{t_0}^{t_1}\int_{B_{R_1}(x_0)} u(t,x)^{\tilde{p}+m-1}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma} \le \tilde{c}\frac{t_1^{\frac{\tilde{p}}{1-m}} - t_0^{\frac{\tilde{p}}{1-m}}}{t_1^{\frac{\tilde{p}+m-1}{1-m}}}\int_{B_{R_1}(x_0)} u(t_1,x)^{\tilde{p}+m-1}\frac{\mathrm{d}x}{|x|^\gamma}. \tag{2.5.8}$$

Finally, letting $s = -\tilde{p}+1-m$, we have $s \in (0, 1-m)$; taking $t^{\tilde{p}/(1-m)} = \left(t_1^{\tilde{p}/(1-m)} + t_0^{\tilde{p}/(1-m)}\right)/2$, we have $2 = \left(t_1^{\tilde{p}/(1-m)} - t_0^{\tilde{p}/(1-m)}\right)/\left(t^{\tilde{p}/(1-m)} - t_0^{\tilde{p}/(1-m)}\right)$, so that (2.5.6), (2.5.7) and (2.5.8) give (2.5.4). Note that we also have

$$\overline{c} \le 2c_2\left[1 + M_p(u_0,\delta,t_0)^{1-m}\right]\left[\frac{h_\sigma(R_1,R_2,x_0)}{(R_1-R_2)^\sigma} + \frac{1}{t_1-t_0}\right] = \overline{c}_1,$$

since we have that $\tilde{p} < 1-m$ implies $t_1 - t = t_1 - \left[\left(t_1^{\tilde{p}/(1-m)} + t_0^{\tilde{p}/(1-m)}\right)/2\right]^{(1-m)/\tilde{p}} \ge (t_1 - t_0)/2$.

• STEP 2. *The finite iteration.* Fix $\varepsilon \in (0, 1-m)$ and assume that $\varepsilon \ne (1-m)(2/r^*)^k$ for all $k \in \mathbb{N}$, to avoid that $(r^*/2)^k\varepsilon = 1-m$ for some $k \in \mathbb{N}$; the remaining cases are similar and will be discussed at the end of the proof. Let k_ε be the smallest positive integer such that $(r^*/2)^{k_\varepsilon}\varepsilon > 1-m$: note that $k_\varepsilon > \log[(1-m)/\varepsilon]/\log[r^*/2]$. We are going to iterate inequality (2.5.4) k_ε times; let us define a decreasing sequence of exponents, and increasing sequences of radii and times for all $0 \le i \le k_\varepsilon$

$$s_i := (r^*/2)^{k_\varepsilon - i}\varepsilon\,, \qquad r_i := R_2 + \frac{i}{k_\varepsilon}(R_1 - R_2) \qquad \text{and} \qquad t_i := T_0 + \frac{i}{k_\varepsilon}(T_1 - T_0)\,. \tag{2.5.9}$$

Note that by construction $s_i \in (0, 1-m)$ for all $i = 0, \dots, k_\varepsilon$, $s_{k_\varepsilon} = \varepsilon$ and $s_0 = (r^*/2)^{k_\varepsilon}\varepsilon > 1-m$, so that we can rewrite inequality (2.5.4) as follows:

$$\|u(t_i)\|_{\mathrm{L}_\gamma^{-s_i}(B_{r_i}(x_0))} \ge c_i \|u(t_{i+1})\|_{\mathrm{L}_\gamma^{-s_{i+1}}(B_{r_{i+1}}(x_0))}, \tag{2.5.10}$$

where $c_i = \underline{c}(s_i, t_i, r_i)$ has the expression

$$\begin{aligned} c_i = &\left[4S_{\gamma,\beta}^2 \left(1 \vee c(m, 1-m-s_{i+1})\left(1 + M_p(u_0, \delta, t_i)^{1-m}\right)\right)\right]^{-\frac{1}{s_{i+1}}} \\ &\times \left[\mu_\gamma(B_{r_i}(x_0))^{-\frac{\sigma}{N-\gamma}} \vee \left(\frac{h_\sigma(r_{i+1}, r_i, x_0)}{(r_{i+1} - r_i)^\sigma} + \frac{1}{t_{i+1} - t_i}\right)\right]^{-\frac{1}{s_{i+1}}} \left[\frac{t_i}{t_{i+1}}\right]^{\frac{1}{1-m}}, \end{aligned} \tag{2.5.11}$$

where $c = c(m, 1-m-s_{i+1})$ is as in (1.1.3). Iterating k_ε-times inequality (2.5.10) we get

$$\|u(t_0)\|_{\mathrm{L}_\gamma^{-s_0}(B_{r_0})} \ge c_0 \|u(t_1)\|_{\mathrm{L}^{s_1}(B_{r_1})} \ge \dots \ge \left(\prod_{i=0}^{k_\varepsilon - 1} c_i\right) \|u(t_{k_\varepsilon})\|_{\mathrm{L}_\gamma^{-s_{k_\varepsilon}}(B_{r_{k_\varepsilon}})}. \tag{2.5.12}$$

Recalling that $t_0 = T_0 < t_{k_\varepsilon} = T_1$, $r_0 = R_2 < r_{k_\varepsilon} = R_1 \le R$, and that $s_0 = (r^*/2)^{k_\varepsilon}\varepsilon > 1-m$ we have

$$\|u(T_0)\|_{\mathrm{L}_\gamma^{-s_0}(B_{R_2})} \ge C \|u(T_1)\|_{\mathrm{L}_\gamma^{-\varepsilon}(B_{R_1})},$$

where $C > 0$ is the lower bound of the product $\prod_{i=0}^{k_\varepsilon - 1} c_i$ that we are going to estimate explicitly below. From formulae (2.5.11) and (2.5.9) we deduce that

$$\begin{aligned} c_i \ge \underline{c}_i &\left(\frac{t_i}{t_{i+1}}\right)^{\frac{1}{1-m}} \left[\mu_\gamma(B_{R_2}(x_0))^{-\frac{\sigma}{N-\gamma}} \vee \left(\frac{h_\sigma(R_1, R_2, x_0)}{(R_1 - R_2)^\sigma} + \frac{1}{T_1 - T_0}\right)\right]^{-\frac{1}{s_{i+1}}} \\ &\times \left[\,\left(1 + M_p(u_0, \delta, T_0)^{1-m}\right)\right]^{-\frac{1}{s_{i+1}}} \end{aligned}$$

since $M_p(u_0, \delta, t_i) \le M_p(u_0, \delta, T_0)$; moreover, $h_\sigma(r_{i+1}, r_i, x_0) \le k_\varepsilon^{(2-\sigma)_+} h_\sigma(R_1, R_2, x_0)$ and $\mu_\gamma(B_{R_2}) \le \mu_\gamma(B_{r_i})$; finally we have set $\underline{c}_i = \left[2^4 S_{\gamma,\beta}^2 c(m, 1-m-s_{i+1})(k_\varepsilon^{2\vee\sigma} + k_\varepsilon)\right]^{\frac{-1}{s_{i+1}}}$. Finally, we can estimate C as follows:

$$\begin{aligned} \prod_{i=0}^{k_\varepsilon - 1} c_i &\ge \left(\frac{T_0}{T_1}\right)^{\frac{1}{1-m}} \left[\prod_{i=0}^{k_\varepsilon - 1} \underline{c}_i\right] \left[\left(1 + M_p(u_0, \delta, T_0)^{1-m}\right) \left(\underline{\mu} \vee \left(\frac{h_\sigma(R_1, R_2, x_0)}{(R_1 - R_2)^\sigma} + \frac{1}{T_1 - T_0}\right)\right)\right]^{\sum_{i=0}^{k_\varepsilon - 1} \frac{-1}{s_{i+1}}} \\ &\ge \underline{\kappa}_4' \left(\frac{T_0}{T_1}\right)^{\frac{1}{1-m}} \left[\left(1 + M_p(u_0, \delta, T_0)^{1-m}\right) \left(\underline{\mu} \vee \left(\frac{h_\sigma(R_1, R_2, x_0)}{(R_1 - R_2)^\sigma} + \frac{1}{T_1 - T_0}\right)\right)\right]^{-\frac{1}{\varepsilon}\frac{1 - \left(\frac{2}{r^*}\right)^{k_\varepsilon}}{1 - \frac{2}{r^*}}}, \end{aligned}$$

where we have put $\underline{\mu} := \mu_\gamma(B_{R_2}(x_0))^{-\frac{\sigma}{N-\gamma}}$ and we have used that $\sum_{i=0}^{k_\varepsilon-1}\frac{1}{s_{i+1}} = \frac{1}{\varepsilon}\sum_{j=0}^{k_\varepsilon-1}\left(\frac{2}{r^*}\right)^j = \frac{1}{\varepsilon}\left[1-\left(\frac{2}{r^*}\right)^{k_\varepsilon}\right]/\left[1-\frac{2}{r^*}\right]$, and that $\prod_{i=0}^{k_\varepsilon-1}(t_i/t_{i+1})^{1/(1-m)} = (T_0/T_1)^{1/(1-m)}$ and we have defined $\underline{\kappa}_4' = \prod_{i=0}^{k_\varepsilon-1}\underline{c}_i > 0$ so that it only depends on $N, m, \beta, \gamma, \varepsilon$.

• *The cases when* $\varepsilon = (1-m)(2/r^*)^k$ *for some* $k \in \mathbb{N}$. When $\varepsilon = (1-m)(2/r^*)^k \in (0, 1-m)$, one can start by a slightly smaller value, say $\tilde{\varepsilon} = \frac{1-m}{2}\left[\left(\frac{2}{r^*}\right)^k + \left(\frac{2}{r^*}\right)^{k+1}\right] < \varepsilon$, proceed as above and obtain (2.5.12) with $k_{\tilde{\varepsilon}} = k+1$, namely $\|u(T_0)\|_{\mathrm{L}_\gamma^{-s_0}(B_{R_2})} \ge C\|u(T_1)\|_{\mathrm{L}_\gamma^{-\tilde{\varepsilon}}(B_{R_1})}$ with $s_0 = (r^*/2)^{k+1}\tilde{\varepsilon} > 1-m$, and then conclude by Hölder's inequality, observing that $\|u(T_1)\|_{\mathrm{L}_\gamma^{-\tilde{\varepsilon}}(B_{R_1})} \ge \mu_\gamma(B_{R_1})^{\frac{1}{\varepsilon}-\frac{1}{\tilde{\varepsilon}}}\|u(T_1)\|_{\mathrm{L}_\gamma^{-\varepsilon}(B_{R_1})}$, since $-\tilde{\varepsilon} > -\varepsilon$; finally we notice that $\frac{1}{\varepsilon}-\frac{1}{\tilde{\varepsilon}} = -\frac{1}{1-m}\left(\frac{r^*}{2}\right)^k\frac{r^*-2}{r^*+2}$. □

2.6 Step 3. Reverse Hölder inequalities

In this subsection we prove the Step 3 of the proof of our positivity result. More precisely we prove $\mathrm{L}^{-\varepsilon} \to \mathrm{L}^{\varepsilon}$ estimates, in the form of reverse Hölder inequalities.

Proposition 2.6.1 (Reverse Hölder inequality for δ-MDP)**.** *Let u_δ be a solution to δ-MDP on $(0,\infty)\times B_{R_0}(x_0)$. Let $u_0\chi_{B_R(x_0)} \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p > p_c$ when $m \in (0, m_c]$ and with $p \ge 1$ when $m \in (m_c, 1)$. Then for all $t \ge t_0 > 0$ and all $0 < R_1 < R$, there exists $\nu_\delta = \nu_\delta(t_0, u_0) > 0$ such that*

$$\|u_\delta(t)\|_{\mathrm{L}^\varepsilon_\gamma(B_{R_1}(x_0))} \le \overline{\kappa}_7^{2/\varepsilon}\mu_\gamma(B_{R_1}(x_0))^{2/\varepsilon}\|u_\delta(t)\|_{\mathrm{L}^{-\varepsilon}_\gamma(B_{R_1}(x_0))} \qquad \text{for all } 0 < \varepsilon < \nu_\delta, \tag{2.6.1}$$

where

$$\nu_\delta := \frac{1}{\overline{\kappa}_{14}\,\overline{\kappa}_6}\left[1 + \frac{R^\sigma}{t_0}\left(\frac{|x_0|}{R}\vee 1\right)^{\beta-\gamma} M_p(u_0,\delta,t_0)^{1-m}\right]^{-1/2},$$

M_p is given in (2.3.1)*, $\overline{\kappa}_{14}$ is as in Lemma 2.6.2, and $\overline{\kappa}_6, \overline{\kappa}_7$ are as in Corollary 0.0.14.*

Remark. As it happens in the elliptic case, the above reverse Hölder inequality plays a fundamental role in the proof of the lower bounds; the above Proposition can be considered the parabolic analogue of the celebrated John-Nirenberg Lemma, cf [100]. As far as we know, in the literature of parabolic equations, there are basically only two techniques that allow one to prove the above estimates: one is due to Moser [73], the other is due to Bombieri and Giusti [101], see also [74]. None of the previous techniques applies directly to our nonlinear setting: in order to ensure the validity of the reverse Hölder inequalities of Corollary 0.0.14, we need to show that $\log u \in BMO_\gamma$ (u is a solution to the MDP or to the δ-MDP). In order to obtain a quantitative control on the BMO_γ norm, we will use the Caccioppoli inequalities (1.1.5) of Lemma 1.1.2, combined with the weighted Poincaré inequality of Proposition 0.0.11 as follows.

Lemma 2.6.2. *Let u_δ be a non-negative solution to δ-MDP on $(0,\infty)\times\Omega$ and let $B_{2R}(x_0) \subset \Omega$. Let $u_0 \in \mathrm{L}^p_\gamma(\Omega)$ for $p > p_c$ if $0 < m \le m_c$ or $p \ge 1$ if $m_c < m < 1$. Then for any $t > 0$ the function $\log u_\delta(t) \in BMO_\gamma(B_R(x_0))$, more precisely there exists a constant $\overline{\kappa}_{14} = \overline{\kappa}_{14}(N, m\gamma, \beta) > 0$ such that for any $t > 0$*

$$\|\log u_\delta(t)\|_{BMO_\gamma(B_R(x_0))} \le \overline{\kappa}_{14}\left[1 + \frac{R^\sigma}{t}\left(\frac{|x_0|}{R}\vee 1\right)^{\beta-\gamma} M_p(u_0,\delta,t)^{1-m}\right]^{\frac{1}{2}} = \frac{1}{\overline{\kappa}_6\nu_\delta}, \tag{2.6.2}$$

where M_p is given in (2.3.1) *and $\overline{\kappa}_6$ is as in Corollary 0.0.14.*

Proof of Lemma 2.6.2. We will write $u = u_\delta$, since no confusion arises here. Let $R > \rho > 0$, $h > 0$ and $\psi \in C_c^\infty(B_{2\rho}(x_0))$. Then by Caccipoli's inequality (1.1.5) of Lemma 1.1.2, we get

$$\begin{aligned}&\frac{m^2(1-m)}{2}\int_{B_{2\rho}(x_0)}\frac{1}{h}\int_\tau^{\tau+h}\psi^2|\nabla\log u|^2\,\mathrm{d}t|x|^{-\beta}\,\mathrm{d}x\\&\le 2(1-m)\int_{B_{2\rho}(x_0)}|\nabla\psi|^2|x|^{-\beta}\,\mathrm{d}x+\int_{B_{2\rho}(x_0)}\frac{u^{1-m}(\tau+h,x)-u^{1-m}(\tau,x)}{h}\psi^2|x|^{-\gamma}\,\mathrm{d}x.\end{aligned}$$

By Lebesgue's Differentiation Theorem, the Steklov averages $\frac{1}{h}\int_s^{s+h}\psi^2|\nabla\log u|^2\,\mathrm{d}t$ converge, for almost every t, to $\psi^2|\nabla\log u|^2$ as $h\to 0$. Using the time monotonicity property of u, namely that $u(\tau+h,x)^{1-m}\le u(\tau,x)^{1-m}\left(\frac{\tau+h}{\tau}\right)$, we get

$$\int_{B_{2\rho}(x_0)}\frac{u^{1-m}(\tau+h,x)-u^{1-m}(\tau,x)}{h}\psi^2|x|^{-\gamma}\,\mathrm{d}x\le\frac{1}{\tau}\int_{B_{2\rho}(x_0)}u^{1-m}(\tau,x)\psi^2|x|^{-\gamma}\,\mathrm{d}x.$$

Now we can take $\psi = 1$ on $B_\rho(y)$ and $\psi = 0$ outside $B_{2\rho}(x_0)$, such that $|\nabla\psi|\le c_N\rho^{-1}$ and let $v = \log u$, letting $h\to 0$ we obtain

$$\begin{aligned}\int_{B_{2\rho(x_0)}}\psi^2|\nabla v(\tau)|^2\frac{\mathrm{d}x}{|x|^\beta}&\le\frac{4}{m^2}\int_{B_{2\rho(x_0)}}|\nabla\psi|^2\frac{\mathrm{d}x}{|x|^\beta}+\frac{2}{m^2(1-m)\tau}\int_{B_{2\rho(x_0)}}u^{1-m}(\tau)\psi^2\frac{\mathrm{d}x}{|x|^\gamma}\\&\le\frac{4c_N\mu_\beta(B_{2\rho}(x_0))}{\rho^2m^2}+\frac{2\mu_\gamma(B_{2\rho}(x_0))}{\tau(1-m)m^2}M_p(u_0,\delta,\tau)^{1-m}.\end{aligned}\tag{2.6.3}$$

In order to estimate the BMO_γ norm of $v=\log u$ on $B_R(x_0)$, we need to estimate the quantity $\mu_\gamma(B_\rho(y))^{-1}\int_{B_\rho(y)}|v-\overline{v}_{B_\rho(y)}||x|^{-\gamma}\,\mathrm{d}x$ on any ball $B_\rho(y)\Subset B_R(x_0)$. To this end, we use the weighted Poincaré inequality (0.0.29), Hölder's inequality and estimate (2.6.3), and we obtain the following:

$$\begin{aligned}\left(\frac{1}{\mu_\gamma(B_\rho(y))}\int_{B_\rho(y)}|v(\tau)-\overline{v(\tau)}_{B_\rho(y)}|\,|x|^{-\gamma}\,\mathrm{d}x\right)^2&\le\frac{P_{\gamma,\beta}^2\,\rho^2}{\mu_\beta(B_\rho(y))}\int_{B_{2\rho}(y)}\psi^2|\nabla v(\tau)|^2|x|^{-\beta}\,\mathrm{d}x\,,\\&\le\frac{4c_NP_{\gamma,\beta}^2D_\beta}{m^2}+\frac{2P_{\gamma,\beta}^2D_\gamma}{m^2(1-m)}\frac{\rho^2\mu_\gamma(B_\rho(y))}{\mu_\beta(B_\rho(y))}\frac{M_p(u_0,\delta,\tau)^{1-m}}{\tau},\end{aligned}$$

where $D_\gamma(D_\beta)$ is the doubling constant of the measure $\mu_\gamma,(\mu_\beta)$ respectively, defined in (0.0.28). Finally, by Lemma 3.4.1

$$\begin{aligned}\frac{\rho^2\mu_\gamma(B_\rho(y))}{\mu_\beta(B_\rho(y))}&\le\overline{\kappa}_{16}\left(\int_{B_\rho(x_0)}|x|^{(\beta-\gamma)\frac{N}{2}}\,\mathrm{d}x\right)^{\frac{2}{N}}\\&\le\overline{\kappa}_{16}\left(\int_{B_R(x_0)}|x|^{(\beta-\gamma)\frac{N}{2}}\,\mathrm{d}x\right)^{\frac{2}{N}}\le c_1\left(\frac{|x_0|}{R}\vee 1\right)^{\beta-\gamma}R^\sigma\,,\end{aligned}$$

where $\overline{\kappa}_{16},c_1>0$ only depend on N,γ,β and $\overline{\kappa}_{16}$ given in 0.0.19. This concludes the proof. □

Proof of Proposition 2.6.1. We have shown that $\log u_\delta(t)\in BMO_\gamma(B_{R_1}(x_0))$ for any $t\ge t_0>0$ in Lemma 2.6.2, more precisely, inequality (2.6.2) gives $\|\log u_\delta(t)\|_{BMO_\gamma(B_{R_1}(x_0))}\le 1/\overline{\kappa}_6\nu_\delta$ for all $t\ge t_0$. We are now in the position to apply inequality (0.0.31) of Corollary 0.0.14, which gives inequality (2.6.1), namely

$$\|u(t)\|_{\mathrm{L}^s_\gamma(B_{R_1}(x_0))}\le\overline{\kappa}_7^{2/s}\mu_\gamma(B_{R_1}(x_0))^{2/s}\|u(t)\|_{\mathrm{L}^{-s}_\gamma(B_{R_1}(x_0))}\,,$$

for all $t \geq t_0 > 0$, and all $0 < s < \nu_\delta \leq 1/\big(\overline{\kappa}_6 \| \log u(t)\|_{BMO_\gamma(B_{R_1}(x_0))}\big)$. Letting $s = \varepsilon$ concludes the proof. □

The above results extend to the case of the MDP as in the following lemma.

Proposition 2.6.3 (Reverse Hölder inequality for MDP). *Let u be a solution to MDP defined on $(0,\infty) \times B_{R_0}(x_0)$, corresponding to the initial datum $u_0\chi_{B_R(x_0)} \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p > p_c$ when $m \in (0, m_c]$ and with $p \geq 1$ when $m \in (m_c, 1)$, with $4R \leq R_0$ and assume that $B_{R_0}(x_0)$ satisfies either (1), (2) or (3); let $T = T(u_0)$ be its extinction time. Then, estimate* (2.6.2) *holds for u, with $\delta = 0$. Moreover, let H_p be as in* (0.0.23)*, then for every $\tau_* \in (0,1]$ we define*

$$\nu_0 := \frac{\tau_*^{\sigma p \vartheta_p}}{\overline{\kappa}_6 \overline{\kappa}_{15}} \left[1 + \left(\frac{|x_0|}{R} \vee 1 \right)^{\beta-\gamma} H_p(u_0, x_0, R)^{1-m} \right]^{-\frac{1}{2}} > 0\,, \tag{2.6.4}$$

so that for every $t \in [\tau_* t_*, t_*] \subset (0,T)$ with $t_* = t_*(u_0, x_0, R)$ is as in (0.0.24)*. We have*

$$\| \log u(t)\|_{BMO_\gamma(B_R(x_0))} \leq (\overline{\kappa}_6 \nu_0)^{-1}\,. \tag{2.6.5}$$

Finally, for all $t \in [\tau_* t_*, t_*] \subset (0,T)$ and all $0 < R_1 < R$, we have

$$\|u(t)\|_{\mathrm{L}^\varepsilon_\gamma(B_{R_1}(x_0))} \leq \overline{\kappa}_7^{2/\varepsilon} \mu_\gamma(B_{R_1}(x_0))^{2/\varepsilon} \|u(t)\|_{\mathrm{L}^{-\varepsilon}_\gamma(B_{R_1}(x_0))} \qquad \textit{for all } 0 < \varepsilon < \nu_0. \tag{2.6.6}$$

The constant $\overline{\kappa}_{15} = \overline{\kappa}'_{15}[m(1-m)]^{-1}$ depends on N, m, γ, β and $\overline{\kappa}_6, \overline{\kappa}_7 > 0$ are as in Corollary 0.0.14.

Proof. Inequality (2.6.5) follows by letting $\delta \to 0$ in inequality (2.6.2), exploiting the lower semi-continuity of the $BMO_\gamma(B_R(x_0))$-norm, then substituting $t = \tau_* t_* < T$, with t_* given in (0.0.24), and finally noticing that

$$\frac{1}{\tau_* t_*} M_p(u_0, \delta, \tau_* t_*)^{1-m} \xrightarrow{\delta \to 0^+} \frac{c_3}{R^\sigma \tau_*^{\sigma p \vartheta_p}} H_p(u_0, x_0, R)^{1-m}\,,$$

where $c_3 > 0$ depends on N, γ, β and m. Inequality (2.6.6) then follows as in the proof of Proposition 2.6.1. □

Remark 2.6.4. When we are in the good fast diffusion range, i.e $m \in (m_c, 1)$, we can choose ν_0 independent of u_0, indeed, by letting $p = 1$ in (2.6.4) and recalling that $H_1(u_0, x_0, R) = \mu_\gamma(B_R(x_0))^{\sigma\vartheta_1} R^{-\sigma(N-\gamma)\vartheta_1}$, we have that

$$\nu_0 := \frac{\tau_*^{\sigma \vartheta_1}}{\overline{\kappa}_6 \overline{\kappa}_{15}} \left[1 + \left(\frac{|x_0|}{R} \vee 1 \right)^{\beta-\gamma} \left(\frac{\mu_\gamma(B_R(x_0))^{\sigma\vartheta_p}}{R^{\sigma(N-\gamma)\vartheta_p}} \right)^{1-m} \right]^{-\frac{1}{2}}.$$

This will have important consequences, but in particular we immediately obtain an *absolute bound of the BMO norm of* $\log u$ on intrinsic cylinders, namely $\| \log u(t)\|_{BMO_\gamma(B_R(x_0))} \leq (\overline{\kappa}_6 \nu_0)^{-1}$, for all $t \in [\tau_* t_*, t_*] \subset (0,T)$. Unfortunately the dependence on u_0 cannot be dropped in the very fast diffusion range, i.e. when $m \in (0, m_c]$.

2.7 End of Step 3 and $\mathrm{L}^{-\infty}-\mathrm{L}^{\varepsilon}$ estimates for MDP

We now sum up all the results of the first three Steps to prove the $\mathrm{L}^{-\infty}-\mathrm{L}^{\varepsilon}$ estimates for the δ-MDP. Next we prove analogous estimates for the MDP by letting $\delta\to 0$.

Proof of Proposition 2.3.1 . Let us first fix $\varepsilon>0$ as in (2.3.2), namely such that

$$0<\varepsilon<\nu_\delta\wedge(1-m)\qquad\text{with}\qquad \nu_\delta=\frac{1}{\overline{\kappa}_{14}\,\overline{\kappa}_6}\left[1+\frac{R_1^\sigma}{t_0}\left(\frac{|x_0|}{R_1}\vee 1\right)^{\beta-\gamma}M_p(u_0,\delta,t_0)^{1-m}\right]^{-1/2},$$

where M_p is given in (2.3.1), $\overline{\kappa}_{14}$ is as in Lemma 2.6.2, so that the Reverse Hölder inequality (2.6.1) holds. Then we are in the position to use the $L^{-s_\varepsilon}-\mathrm{L}^{-\varepsilon}$ smoothing effect of Proposition 2.5.1 with $\varepsilon\in(0,1-m)$ as above , together with the $\mathrm{L}^{-\infty}-\mathrm{L}^{-s_\varepsilon}$ lower bounds of Proposition 2.4.1 ; combining all the above results we obtain, choosing $\overline{R}=(R_1+R_2)/2$ and times as in the statement:

$$\begin{aligned}
\inf_{(T_1,T_2]\times B_{R_2}(x_0)}u &\ge \underline{\kappa}_3\left[\left(1+M_p(u_0,\delta,T_0)^{1-m}\right)\left(\frac{h_\sigma\left(\overline{R},R_2,x_0\right)}{\left(\overline{R}-R_2\right)^\sigma}+\frac{1}{T_1-T_0}\right)\right]^{\eta_\varepsilon}\\
&\times\left[\left(\frac{T_0}{T_2}\right)^{\frac{1}{1-m}}\left(\int_{B_{\overline{R}}(x_0)}u^{-s}(T_2,x)|x|^{-\gamma}\,\mathrm{d}x\right)^{-\frac{1}{s_\varepsilon}}\right]^{\frac{s_\varepsilon}{s_\varepsilon+m-1}}(T_2-T_0)^{-\frac{1}{s_\varepsilon+m-1}}\\
&\ge \underline{\kappa}_3\underline{\kappa}_4^{\frac{s_\varepsilon}{s_\varepsilon+m-1}}\left[2^{(2\vee\sigma)}\left(1+M_p(u_0,\delta,T_0)^{1-m}\right)\left(\underline{\mu}\vee\left(\frac{h_\sigma(R_1,R_2,x_0)}{(R_1-R_2)^\sigma}+\frac{1}{T_1-T_0}\right)\right)\right]^{\eta_\varepsilon+\frac{s_\varepsilon}{s_\varepsilon+m-1}\zeta_\varepsilon}\\
&\times\left[\left(\frac{T_0}{T_2}\right)^{\frac{1}{1-m}}\left(\frac{T_2}{T_3}\right)^{\frac{1}{1-m}}\left(\int_{B_{R_1}(x_0)}u_\delta(T_3,x)^{-\varepsilon}|x|^{-\gamma}\,\mathrm{d}x\right)^{-\frac{1}{\varepsilon}}\right]^{\frac{s_\varepsilon}{s_\varepsilon+m-1}}(T_2-T_0)^{-\frac{1}{s_\varepsilon+m-1}}\\
&\ge \underline{\kappa}_3\underline{\kappa}_4^{\frac{s_\varepsilon}{s_\varepsilon+m-1}}\left[2^{(2\vee\sigma)}\left(1+M_p(u_0,\delta,T_0)^{1-m}\right)\left(\underline{\mu}\vee\left(\frac{h_\sigma(R_1,R_2,x_0)}{(R_1-R_2)^\sigma}+\frac{1}{T_1-T_0}\right)\right)\right]^{\eta_\varepsilon+\frac{s_\varepsilon}{s_\varepsilon+m-1}\zeta_\varepsilon}\\
&\times\left[\left(\frac{T_0}{T_3}\right)^{\frac{1}{1-m}}\overline{\kappa}_7^{-\frac{2}{s_\varepsilon}}\mu_\gamma(B_{R_1}(x_0))^{-\frac{2}{s_\varepsilon}}\left(\int_{B_{R_1}(x_0)}u_\delta(T_3,x)^{\varepsilon}|x|^{-\gamma}\,\mathrm{d}x\right)^{\frac{1}{\varepsilon}}\right]^{\frac{s_\varepsilon}{s_\varepsilon+m-1}}(T_2-T_0)^{-\frac{1}{s_\varepsilon+m-1}}
\end{aligned}$$

where we have put $\underline{\mu}:=\mu_\gamma(B_{R_2}(x_0))^{-\frac{\sigma}{N-\gamma}}$. In the first step we have used (2.4.1) with $s_\varepsilon>1-m$ as in (2.5.2), and $0<R_2<\overline{R}<R_1$ and $T_1\in(T_0,T_2)$. Note that $\eta_\varepsilon:=-\frac{1}{s_\varepsilon+m-1}\left(\frac{N-\gamma}{2+\beta-\gamma}+1\right)$, $\zeta_\varepsilon=-\frac{1}{\varepsilon}\frac{1-\left(\frac{2}{r^*}\right)^{k_\varepsilon}}{1-\frac{2}{r^*}}$; h_σ and M_p are defined in (1.1.1) and (2.3.1) respectively, and $\underline{\kappa}_3>0$ is as in (2.4.1) only depending on $s,\widetilde{p},N,\gamma,\beta,m$. In the second step we have used (2.5.1) with $\underline{\kappa}_4$ as in (2.5.2), noticing that $M_p(u_0,\delta,T_2)\le M_p(u_0,\delta,T_0)$ and that $h_\sigma\left(\overline{R},R_2,x_0\right)\asymp h_\sigma\left(R_1,\overline{R},x_0\right)\le 2^{(2-\sigma)_+}h_\sigma\left(R_1,R_2,x_0\right)$, and where $s_\varepsilon>1-m$ and $\underline{\kappa}_4>0$ are given in (2.5.2). In the third step we have used the Reverse Hölder inequality (2.6.1) with ε and ν_δ as above, with $\overline{\kappa}_7$ as in Corollary 0.0.14. □

Taking the limit $\delta\to 0$. $\mathrm{L}^{-\infty}-\mathrm{L}^{\varepsilon}$ interior estimates for MDP. Consider the solution u of the MDP with initial data u_0. Then the solutions of the "lifted problem" δ-MDP u_δ are ordered with respect to δ: more precisely, for $\delta>\delta'$, for any $x\in B_R(x_0)$ and for any $t\in(0,\infty)$

$$u_\delta(t,x)\ge u_{\delta'}(t,x).$$

In particular, for any $x\in B_R(x_0)$ and for any $t\in(0,\infty)$, the limit as $\delta\to 0$ exists and is equal to $u(t,x)$. See [86, Section B.3] for more details. Note that the constants in the inequality (2.3.3) remain stable as $\delta \to 0^+$ (see also Proposition 2.6.3). As an immediate consequence of Proposition 2.3.1 we get the following result.

Corollary 2.7.1 ($\mathrm{L}^{-\infty} - \mathrm{L}^{\varepsilon}$ estimates for MDP)**.** *Let u be a strong (super)solution to MDP on $(0,\infty) \times B_{R_0}(x_0)$, corresponding to the initial datum $u_0\chi_{B_R(x_0)} \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p > p_c$ when $m \in (0, m_c]$ and with $p \ge 1$ when $m \in (m_c, 1)$, moreover assume that $8R = R_0$ and that $B_{R_0}(x_0)$ satisfies either (1), (2) or (3); let $T = T(u_0)$ be the extinction time. Let $\widetilde{H}_p$ be as in (0.0.23), and define, for any fixed $\tau_* \in (0, \frac{1}{3})$,*

$$\nu_0 := \frac{m(1-m)\tau_*^{\sigma p \vartheta_p}}{\overline{\kappa}'_{15} \widetilde{H}_p(u_0, x_0, R)^{\frac{1}{2}}} \in (0, 1-m)\,. \tag{2.7.1}$$

Then, $[2\tau_* \overline{t}_*, (1-\tau_*)\overline{t}_*] \subset [\tau_* \overline{t}_*, \overline{t}_*] \subset (0, T)$ with $\overline{t}_* = \overline{t}_*(u_0, x_0, R)$ is as in (2.8.1). Moreover, for any $\varepsilon \in (0, \nu_0)$ there exist $s_\varepsilon > 1-m$ and $\underline{\kappa}_\varepsilon > 0$ such that

$$\inf_{[2\tau_*\overline{t}_*, (1-\tau_*)\overline{t}_*] \times B_{2R}(x_0)} u \ge \underline{\kappa}_\varepsilon \left(\int_{B_{4R}(x_0)} u(\overline{t}_*, x)^\varepsilon |x|^{-\gamma} \, \mathrm{d}x \right)^{\frac{s_\varepsilon}{\varepsilon(s_\varepsilon + m - 1)}} \tag{2.7.2}$$

with

$$\begin{aligned} \underline{\kappa}_\varepsilon &:= \underline{\kappa}_2 \, \tau_*^{\theta_\varepsilon} \mu_\gamma(B_{4R}(x_0))^{-\frac{2}{s_\varepsilon + m - 1}} \left((1 - 2\tau_*)\overline{t}_* \right)^{-\frac{1}{s_\varepsilon + m - 1}} \\ &\times \left[\widetilde{H}_p(u_0, x_0, R) \left(1 \vee \frac{\overline{t}_*}{R^\sigma} \right) \left(\mu_\gamma(B_{2R}(x_0))^{-\frac{\sigma}{N-\gamma}} \vee \left(\frac{h_\sigma(4R, 2R, x_0)}{(2R)^\sigma} + \frac{1}{\tau_* \overline{t}_*} \right) \right) \right]^{\eta_\varepsilon + \frac{s_\varepsilon \zeta_\varepsilon}{s_\varepsilon + m - 1}} \end{aligned} \tag{2.7.3}$$

where $s_\varepsilon > 0, \eta_\varepsilon, \zeta_\varepsilon < 0$ are as in (2.3.4), $\theta_\varepsilon = (1 - p\sigma\vartheta_p)\left(\eta_\varepsilon + \frac{s_\varepsilon \zeta_\varepsilon}{s_\varepsilon + m - 1}\right) + \frac{1}{1-m} \frac{s_\varepsilon}{s_\varepsilon + m - 1}$; h_σ, M_p and ϑ_p are defined in (1.1.1), (2.3.1) and (0.0.21) respectively; finally, $\underline{\kappa}_2 > 0$ depends only on $N, m, p, \beta, \gamma, \varepsilon$, through $\underline{\kappa}_3, \underline{\kappa}_4$ defined in (2.4.1), (2.5.2), and through $\overline{\kappa}_6, \overline{\kappa}_7$, which are defined in Corollary 0.0.14; $\overline{\kappa}'_{15}$ is the same as in Proposition 2.6.3.

2.8 Step 4. Reverse $\mathrm{L}^1 - \mathrm{L}^\varepsilon$ smoothing effects and interior lower bounds for MDP.

Next we obtain a useful Lemma about quantitative positivity of local L^1_γ norms and a local reverse $\mathrm{L}^\varepsilon - \mathrm{L}^1$ smoothing effects for solutions to the MDP.

Lemma 2.8.1. *Let u be the solution to MDP on $(0,\infty) \times B_{R_0}(x_0)$, corresponding to the initial datum $u_0\chi_{B_R(x_0)} \in \mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p > p_c$ when $m \in (0, m_c]$ and with $p \ge 1$ when $m \in (m_c, 1)$, with $4R \le R_0$ and assume that $B_{R_0}(x_0)$ satisfies either (1), (2) or (3); let $T = T(u_0)$ be its extinction time and define $0 \le \overline{t}_* \le T$ as*

$$\overline{t}_* = \overline{t}_*(u_0, x_0, R) = \frac{\overline{\kappa}'^{-1}_{10}}{2^{2-m}} \frac{R^\sigma}{\mu_\gamma(B_R(x_0))^{1-m}} \|u_0\|^{1-m}_{\mathrm{L}^1_\gamma(B_R(x_0))}, \tag{2.8.1}$$

where $\overline{\kappa}'_{10} \ge 1$ is the constant defined in (3.4.8) depending only on N, γ, β and m. Then there exists $\underline{\kappa}_0 > 0$ such that

$$\frac{\underline{\kappa}_0}{\mu_\gamma(B_R(x_0))} \int_{B_R(x_0)} u_0 |x|^{-\gamma} \, \mathrm{d}x \le \frac{1}{\mu_\gamma(B_{4R}(x_0))} \int_{B_{4R}(x_0)} u\left(\overline{t}_*, x\right) |x|^{-\gamma} \, \mathrm{d}x, \tag{2.8.2}$$

where $\underline{\kappa}_0$ depends on N,γ,β,m. Moreover, for any $\varepsilon\in(0,1)$ the following estimate holds

$$\left(\frac{1}{\mu_\gamma(B_R(x_0))}\int_{B_R(x_0)} u_0|x|^{-\gamma}\,\mathrm{d}x\right)^\varepsilon \le \underline{\kappa}_0^{-1}\underline{\kappa}_9^{1-\varepsilon}\frac{H_p\left(u_0,x_0,R\right)^{1-\varepsilon}}{\mu_\gamma(B_{4R}(x_0))}\int_{B_{4R}(x_0)} u^\varepsilon(\bar t_*,x)|x|^{-\gamma}\,\mathrm{d}x, \quad (2.8.3)$$

where $H_p\left(u_0,x_0,R\right)$ is defined in (0.0.23) , $\underline{\kappa}_9=\overline{\kappa}_{12}\left(\overline{\kappa}_{10}'2^{2-m}\right)^{(N-\gamma)\vartheta_p}\omega_{N,\gamma}^{\sigma\vartheta_p}$, with $\overline{\kappa}_{12}>0$ depending only on N,m,γ,β,p (given in (2.1.1)) and $\omega_{N,\gamma}$ being such that $\omega_{N,\gamma}R^{N-\gamma}=\mu_\gamma(B_R(0))$.

Proof. Let $u(t,x)$ be a solution to MDP over the cylinder $B_{R_0}(x_0)\times(0,T)$. Applying inequality (1.2.2) with times $t=0,\tau=\bar t_*$ and radii R and $2R$ we obtain

$$\begin{aligned}\frac{1}{\mu_\gamma(B_R(x_0))}\int\limits_{B_R(x_0)} u_0\frac{\mathrm{d}x}{|x|^\gamma} &\le \frac{2^{\frac{1}{1-m}}}{\mu_\gamma(B_R(x_0))}\left[\int\limits_{B_{4R}} u(\bar t_*,x)\frac{\mathrm{d}x}{|x|^\gamma}+\frac{\overline{\kappa}_{10}'^{\frac{1}{1-m}}\left(\bar t_*\right)^{\frac{1}{1-m}}}{R^{\frac{\sigma}{1-m}}}\mu_\gamma(B_R(x_0))\right]\\ &\le \frac{2^{\frac{1}{1-m}}D_\gamma^2}{\mu_\gamma(B_{4R}(x_0))}\int\limits_{B_{4R}(x_0)} u(\bar t_*,x)\frac{\mathrm{d}x}{|x|^\gamma}+\frac{1}{2}\frac{1}{\mu_\gamma(B_R)}\int\limits_{B_R(x_0)} u_0\frac{\mathrm{d}x}{|x|^\gamma},\end{aligned} \quad (2.8.4)$$

where we used the fact that u_0 is supported in B_R, the doubling property of the measure μ_γ and the fact that $u(t,x)>0$. Inequality (2.8.2) is then deduced from (2.8.4) with constant $\underline{\kappa}_0=\frac{1}{2}2^{-\frac{1}{1-m}}D_\gamma^{-2}$. We now turn our attention to inequality (2.8.3), which will be deduced from (2.8.2). Let $\varepsilon\in(0,1)$: using the smoothing-effect inequality (2.1.1), namely $\|u(t)\|_{\mathrm{L}^\infty(B_{R_0}(x_0))}\le \overline{\kappa}_{12}\|u_0\|^{\sigma p\vartheta_p}_{\mathrm{L}^p_\gamma(B_R(x_0))}t^{-(N-\gamma)\vartheta_p}$ (recall that $\overline{\kappa}_{12}$ does not depend on $B_R(x_0)$) applied at $t=\bar t_*$ we get

$$\|u(\bar t_*)\|_{\mathrm{L}^\infty(B_{R_0})}\le \underline{\kappa}_9 H_p(u_0,x_0,R)\frac{\|u_0\|_{\mathrm{L}^1_\gamma(B_R(x_0))}}{\mu_\gamma(B_R(x_0))}$$

where in the last step we have used inequality (2.8.2) and the equality $\omega_{N,\gamma}R^{N-\gamma}=\mu_\gamma(B_R(0))$; the constant $\underline{\kappa}_9=\overline{\kappa}_{12}\left(\overline{\kappa}_{10}'2^{2-m}\right)^{(N-\gamma)\theta_p}\omega_{N,\gamma}^{\sigma\vartheta_p}$. We finally combine the above inequality with

$$\|u(\bar t_*)\|_{\mathrm{L}^1_\gamma(B_{4R}(x_0))}\le \|u(\bar t_*)\|^{1-\varepsilon}_{\mathrm{L}^\infty(B_{4R}(x_0))}\|u(\bar t_*)\|^{\varepsilon}_{\mathrm{L}^\varepsilon_\gamma(B_{4R}(x_0))}\,,$$

and we obtain inequality (2.8.3). □

Putting together all the results of the 4 Steps, we obtain the following

Corollary 2.8.2 ($\mathrm{L}^{-\infty}-\mathrm{L}^1$ estimates for MDP). *Let u be a strong (super)solution to MDP on $(0,\infty)\times B_{R_0}(x_0)$, corresponding to $u_0\chi_{B_R(x_0)}\in\mathrm{L}^p_\gamma(B_{R_0}(x_0))$ with $p>p_c$ if $m\in(0,m_c]$ or $p\ge1$ if $m\in(m_c,1)$, moreover assume that $4R=R_0$ and that $B_{R_0}(x_0)$ satisfies either (1), (2) or (3); let $T=T(u_0)$ be its extinction time and define $t_*\in[0,T]$ by*

$$t_*=t_*(u_0,x_0,R)=\kappa_*R^\sigma\frac{\|u_0\|^{1-m}_{\mathrm{L}^1_\gamma(B_R(x_0))}}{\mu_\gamma(B_R(x_0))^{1-m}} \quad (2.8.5)$$

where $\kappa_*=5^{-1}2^m\overline{\kappa}_{10}'^{-1}$, $\overline{\kappa}_{10}'\ge1$ depends only on N,γ,β and m as in (3.4.8). Then, there exists $\underline{\kappa}>0$ such that

$$\inf_{[\frac{t_*}{2},t_*]\times B_{2R}(x_0)} u\ge \underline{\kappa}(\widetilde H_p,R)\,t_*^{\frac{r^*}{(1-m)(r^*-2)}}, \quad (2.8.6)$$

where $\underline{\kappa}$ has an explicit form given in (2.8.10)*, in particular for R bounded and $\widetilde{H}_p >> 1$ we have*

$$\underline{\kappa} \asymp \left(\frac{R^{c_1}}{\widetilde{H}_p^{c_2}}\right)^{\frac{\widetilde{H}_p^{1/2}}{m(1-m)}} \qquad \textit{with} \quad \widetilde{H}_p(u_0,x_0,R) := 1 + \left(\frac{|x_0|}{R} \vee 1\right)^{\beta-\gamma} H_p(u_0,x_0,R)^{1-m} \geq 1\,, \tag{2.8.7}$$

and with $c_1, c_2 > 0$ only depending on N, m, p, β, γ.

Proof. Let H_p be as in (0.0.23) and fix $\overline{\tau}_* = 5^{-1}$, $\kappa_* = (1-\overline{\tau}_*)2^{m-2}\overline{\kappa}_{10}'^{-1}$, and $t_* := (1-\overline{\tau}_*)\overline{t}_*$, where $\overline{t}_*$ is defined in (2.8.1); recall that $D_\gamma \geq 1$ is the doubling constant of the measure μ_γ, and $\overline{\kappa}_{10}' \geq 1$ is as in (3.4.8) (since $\overline{\kappa}_{10}'$ is a constant in an upper bound, hence without loss of generality we can take it bigger than 1). Next we fix ν_0 (that depends on τ_*) as in (2.7.1) with $\tau_* = \overline{\tau}_*$. Finally, we choose $\varepsilon = \varepsilon_0 := (2/r^*)^{k_0}(1-m)$, where k_0 is the smallest integer such that $\varepsilon_0 < \nu_0$. Note that we have $k_0 \geq \log(\nu_0)/\log(2/r^*) - \log(1-m)/\log(2/r^*)$. With these choices, we know that $k_\varepsilon = k_0 + 1$ and the exponents $s_\varepsilon > 1-m, \eta_\varepsilon, \zeta_\varepsilon < 0$ given in (2.3.4) become

$$s_0 := s_{\varepsilon_0} = \left(\frac{r^*}{2}\right)^{k_\varepsilon} \varepsilon_0 = (1-m)\frac{r^*}{2} > 1-m\,, \qquad \frac{s_0}{s_0+m-1} = \frac{r^*}{r^*-2} > 0\,,$$
$$\eta_{s_0} := \eta_{\varepsilon_0} = -2\frac{\frac{N-\gamma}{2+\beta-\gamma}+1}{(1-m)(r^*-2)} < 0\,, \quad \zeta_0 := \zeta_{\varepsilon_0} = -\frac{1}{\varepsilon_0}\frac{r^*(1-m)-2\varepsilon_0}{(1-m)(r^*-2)} < 0\,.$$

Note that, even if ε_0 depends on ν_0 (through k_0), the exponents s_ε and η_ε now only depend on N, γ, β, m. We are now in the position to combine inequalities (2.7.2) and (2.8.3) (with $R_0 \geq 4R$) as follows: for any $0 < R_1 < 4R \leq R_0$ we have

$$\begin{aligned}
\inf_{[2\overline{\tau}_*\overline{t}_*,(1-\overline{\tau}_*)\overline{t}_*]\times B_{2R}(x_0)} u &\geq \underline{\kappa}_{\varepsilon_0}\left(\int_{B_{4R}(x_0)} u(\overline{t}_*,x)^{\varepsilon_0}|x|^{-\gamma}\,\mathrm{d}x\right)^{\frac{s_0}{\varepsilon_0(s_0+m-1)}}\\
&\geq \underline{\kappa}_{\varepsilon_0}\left(\frac{\mu_\gamma(B_{4R}(x_0))}{\underline{\kappa}_9 H_p\left(u_0,x_0,R\right)^{1-\varepsilon_0}}\right)^{\frac{s_0}{\varepsilon_0(s_0+m-1)}}\left(\frac{1}{\mu_\gamma(B_R(x_0))}\int_{B_R(x_0)} u_0|x|^{-\gamma}\,\mathrm{d}x\right)^{\frac{s_0}{s_0+m-1}}\\
&\geq \underline{\kappa}(\widetilde{H}_p,R)\,t_*^{\frac{s_0}{(1-m)(s_0+m-1)}} = \underline{\kappa}(\widetilde{H}_p,R)\,t_*^{\frac{r^*}{(1-m)(r^*-2)}}
\end{aligned}$$

where in the second step we have used inequality (2.8.3) and $\underline{\kappa}_\varepsilon$ is as in (2.7.3) and $\underline{\kappa}_9$ is as in (2.8.3). Finally we have used the expression of t_* given in (2.8.5). We estimate the $\underline{\kappa}(\widetilde{H}_p,R)$ as follows

$$\begin{aligned}
&\underline{\kappa}_{\varepsilon_0}\left(\frac{\mu_\gamma(B_{4R}(x_0))}{\underline{\kappa}_9 H_p\left(u_0,x_0,R\right)^{1-\varepsilon_0}}\right)^{\frac{s_0}{\varepsilon_0(s_0+m-1)}} \qquad\qquad (2.8.8)\\
&\geq \underline{\kappa}_2\,\overline{\tau}_*^{\theta_\varepsilon}\mu_\gamma(B_{4R}(x_0))^{\frac{s_0-2\varepsilon_0}{\varepsilon_0(s_0+m-1)}}\left((1-2\overline{\tau}_*)\overline{t}_*\right)^{-\frac{1}{s_0+m-1}}\widetilde{H}_p\left(u_0,x_0,R\right)^{\frac{(\varepsilon_0-1)s_0}{\varepsilon_0(s_0+m-1)}}\underline{\kappa}_9^{-\frac{s_0}{\varepsilon_0(s_0+m-1)}}\\
&\times\left[\widetilde{H}_p(u_0,x_0,R)\left(1\vee\frac{\overline{t}_*}{R^\sigma}\right)\left(\mu_\gamma(B_{2R}(x_0))^{-\frac{\sigma}{N-\gamma}}\vee\left(\frac{h_\sigma(4R,2R,x_0)}{(2R)^\sigma}+\frac{1}{\overline{\tau}_*\overline{t}_*}\right)\right)\right]^{\eta_{s_0}+\frac{s_0\,\zeta_0}{s_0+m-1}},
\end{aligned}$$

where we just used the expression of $\widetilde{H}_p = \widetilde{H}_p(u_0,x_0,R)$ given in (2.8.7) and rewritten in the constant appearing in (2.7.3). We then estimate

$$
\begin{aligned}
&\left[\widetilde{H}_p(u_0,x_0,R)\left(1\vee\frac{\overline{t}_*}{R^\sigma}\right)\left(\mu_\gamma(B_{2R}(x_0))^{-\frac{\sigma}{N-\gamma}}\vee\left(\frac{h_\sigma(4R,2R,x_0)}{(2R)^\sigma}+\frac{1}{\overline{\tau}_*\overline{t}_*}\right)\right)\right]^{\eta_{s_0}+\frac{s_0\,\zeta_0}{s_0+m-1}} \qquad (2.8.9)\\
&\ge \widetilde{H}_p(u_0,x_0,R)^{\eta_{s_0}+\frac{s_0\,\zeta_0}{s_0+m-1}}\left[\left(1\vee\frac{t_*}{(1-\overline{\tau}_*)R^\sigma}\right)\times\right.\\
&\quad\times\left.\left(\frac{h_\sigma(4R,2R,x_0)}{(2R)^\sigma}\vee\mu_\gamma(B_{2R}(x_0))^{-\frac{\sigma}{N-\gamma}}\right)\left(1+\frac{(2R)^\sigma(1-\overline{\tau}_*)}{\overline{\tau}_*t_*}\right)\right]^{\eta_{s_0}+\frac{s_0\zeta_0}{s_0+m-1}},
\end{aligned}
$$

where we just used the expression of $\overline{t}_*$ given in the beginning of the proof. Recall that the expression of $\underline{\kappa}_2$, given in Proposition 2.3.1 is:

$$
\underline{\kappa}_2=\underline{\kappa}_3\underline{\kappa}_4^{\frac{s_0}{s_0+m-1}}\overline{\kappa}_7^{-\frac{2}{s_\varepsilon+m-1}}2^{(2\vee\sigma)(\eta_{s_0}+\frac{s_0}{s_0+m-1}\zeta_0)},
$$

where $\underline{\kappa}_4=\underline{\kappa}_4'\mu_\gamma(B_{4R})^{-\frac{1}{\varepsilon_0}\frac{r^*-2}{r^*+2}}$. All the above estimates finally give the expression of $\underline{\kappa}$

$$
\begin{aligned}
\underline{\kappa}:=&\underline{\kappa}_3\underline{\kappa}_4'^{\frac{s_0}{s_0+m-1}}\overline{\kappa}_7^{-\frac{2}{s_\varepsilon+m-1}}2^{(2\vee\sigma)(\eta_{s_0}+\frac{s_0}{s_0+m-1}\zeta_0)}\underline{\kappa}_9^{-\frac{s_0}{\varepsilon_0(s_0+m-1)}} \qquad (2.8.10)\\
&\times\overline{\tau}_*^{\theta_\varepsilon}\mu_\gamma(B_{4R}(x_0))^{-\frac{2-\frac{s_0}{\varepsilon_0}(\frac{2r^*}{r^*+2})}{s_0+m-1}}\widetilde{H}_p^{\eta_{s_0}+\frac{s_0}{s_0+m-1}(\zeta_0+\frac{\varepsilon_0-1}{\varepsilon_0})}\left(\frac{1-2\overline{\tau}_*}{1-\overline{\tau}_*}t_*\right)^{-\frac{1}{s_0+m-1}}\\
&\times\left[\left(1\vee\frac{t_*}{(1-\overline{\tau}_*)R^\sigma}\right)\left(\frac{h_\sigma(4R,2R,x_0)}{(2R)^\sigma}\vee\mu_\gamma(B_{2R}(x_0))^{-\frac{\sigma}{N-\gamma}}\right)\left(1+\frac{(2R)^\sigma(1-\overline{\tau}_*)}{\overline{\tau}_*t_*}\right)\right]^{\eta_{s_0}+\frac{s_0\zeta_0}{s_0+m-1}}.
\end{aligned}
$$

We also recall that as in Lemma 2.8.1, $\underline{\kappa}_9=\overline{\kappa}_{12}\left(\overline{\kappa}_{10}'2^{2-m}\right)^{(N-\gamma)\vartheta_p}\omega_{N,\gamma}^{\sigma\vartheta_p}$, with $\overline{\kappa}_{12}>0$ that depends only on N,m,γ,β,p and is given in (2.1.1) and $\omega_{N,\gamma}$ being such that $\omega_{N,\gamma}R^{N-\gamma}=\mu_\gamma(B_R(0))$. Observe that for sufficiently small $\varepsilon_0<\nu_0\sim m(1-m)/\widetilde{H}_p^{1/2}$, we have that $\zeta_0+\frac{\varepsilon_0-1}{\varepsilon_0}\sim\frac{c}{\varepsilon_0}$. Finally, note that when R is bounded and H_p is large enough, we have that $\underline{\kappa}\sim\left(R^{c_4}/\widetilde{H}_p^{c_5}\right)^{\widetilde{H}_p^{1/2}/m(1-m)}$, where $c_i>0$ only depend on N,m,p,β,γ. □

2.9 Positivity for local solutions.

We are now in the position to conclude the proof of the main results of this Part, Theorems 0.0.4 and 2.0.1.

End of the proof Theorem 2.0.1. Let $u(t,x)$ be a solution to the MDP on the cylinder $B_{4R}(x_0)\times(0,T)$, where $T=T(u_0)$ is the extinction time. Recall that $0\le u_0\in\mathrm{L}^p_\gamma(B_R(x_0))$ for $p>p_c$ if $0<m\le m_c$ or $p\ge1$ if $m_c<m<1$. Let $M=\int_{B_R(x_0)}u_0|x|^{-\gamma}\,\mathrm{d}x>0$ and define the rescaled solution $\hat{u}$ as follows

$$
\hat{u}\left(\hat{t},\hat{x}\right)=\frac{R^{N-\gamma}}{M}u\left(\tau\hat{t},R\hat{x}\right),\qquad \tau=R^{\sigma-(N-\gamma)(1-m)}M^{1-m}.
$$

The rescaled solution $\hat{u}$ solves the MDP on the cylinder $B_4(R^{-1}x_0)\times(0,\hat{T})$ with mass 1 and extinction time $\hat{T}$. We are in the position to apply Corollary 2.8.2 to get: (recall that $\hat{x}=R^{-1}x$ and $\hat{t}=\tau^{-1}t$)

$$
\inf_{x\in B_2(\hat{x}_0)}\hat{u}(\hat{t}_*,\hat{x})\ge\underline{\kappa}(\widetilde{H}_p,1)\,\hat{t}_*^{\frac{r^*}{(1-m)(r^*-2)}},\qquad\text{where}\qquad \hat{t}_*=\kappa_*\frac{1}{\mu_\gamma(B_1(\hat{x}_0))^{1-m}} \qquad (2.9.1)
$$

where the value of $\underline{\kappa}(\widetilde{H}_p, 1)$ is given in (2.8.10), while $\widetilde{H}_p(\hat{u}_0, \hat{x}_0, 1)$ and κ_* are given in Corollary 2.8.2. Note that the quantity $\widetilde{H}_p$ is actually scaling invariant, namely

$$\widetilde{H}_p(\hat{u}_0, \hat{x}_0, 1) = \widetilde{H}_p(u_0, x_0, R). \tag{2.9.2}$$

Note also that $\hat{t}_*$ only depends on N, m, γ, β, but not on u, u_0 nor R, x_0 (here is where we use either assumption (1), (2) or (3)); indeed, using $\mu_\gamma(B_1(\rho^{-1}x_0)) = \rho^{\gamma-N}\mu_\gamma(B_\rho(x_0))$ it is straightforward to check that $\hat{t} \asymp \kappa_*$. Recalling that $\hat{t} \mapsto \hat{t}^{-\frac{1}{1-m}}\hat{u}(\hat{t}, \hat{x})$ is non-increasing in time for almost every $\hat{x} \in B_1(\hat{x}_0)$, we get as a consequence of (2.9.1), for all $0 \le \hat{t} \le \hat{t}_*$

$$\inf_{\hat{x}\in B_2(\hat{x}_0)} \hat{u}(\hat{t}, \hat{x}) \ge \left(\frac{\hat{t}}{\hat{t}_*}\right)^{\frac{1}{1-m}} \inf_{\hat{x}\in B_2(\hat{x}_0)} \hat{u}(\hat{t}_*, \hat{x}) \ge \underline{\kappa}(\widetilde{H}_p, 1)\, \hat{t}_*^{\frac{2}{(1-m)(r^*-2)}}\, \hat{t}^{\frac{1}{1-m}} := \hat{\underline{\kappa}}(\widetilde{H}_p, 1)\, \hat{t}^{\frac{1}{1-m}}\,. \tag{2.9.3}$$

We have used a scaling argument to obtain a cleaner constant $\underline{\kappa}$ in the final lower bound (2.0.2), in this way, $\underline{\kappa} = \underline{\kappa}(\widetilde{H}_p, 1)$ shall depend on R, x_0 only through H_p. This is a consequence of our assumptions (1),(2) or (3) and the explicit expression of $\hat{\underline{\kappa}}(\widetilde{H}_p, 1)$ given in (2.8.10) (recall that $\overline{\tau}_* = 1/5$):

$$\begin{aligned}\underline{\kappa} &:= \underline{\kappa}_3 \underline{\kappa}_4'^{\frac{s_0}{s_0+m-1}}\, \overline{\kappa}_7^{-\frac{2}{s_\varepsilon+m-1}}\, 2^{(2\vee\sigma)(\eta_{s_0}+\frac{s_0}{s_0+m-1}\zeta_0)}\, \underline{\kappa}_9^{-\frac{s_0}{\varepsilon_0(s_0+m-1)}}\, 5^{-\theta_{\varepsilon_0}}\, \left(3\hat{t}_*\right)^{-\frac{1}{s_0+m-1}} \\ &\times C_{N,\gamma}^{-\frac{2-\frac{s_0}{\varepsilon_0}(\frac{2r^*}{r^*+2})}{s_0+m-1}}\, \widetilde{H}_p^{\eta_{s_0}+\frac{s_0}{s_0+m-1}(\zeta_0+\frac{\varepsilon_0-1}{\varepsilon_0})} \left[\left(1 \vee \frac{5}{4}\hat{t}_*\right) C_{\gamma,\beta}\left(1 + \frac{4}{\hat{t}_*}\right)\right]^{\eta_{s_0}+\frac{s_0\zeta_0}{s_0+m-1}}\end{aligned} \tag{2.9.4}$$

In the computation of the above constant, we have used systematically the identity $\mu_\gamma(B_1(\rho^{-1}x_0) = \rho^{\gamma-N}\mu_\gamma(B_\rho(x_0))$ which holds under our assumptions: as a consequence all the constants in the right-hand side of formula (2.8.10) will depend only on N, m, γ, β, and some of them on H_p. More precisely, $\underline{\kappa}_9 \asymp \underline{\kappa}_0^{-1}\overline{\kappa}_{12}$ as well as $\mu_\gamma(B_4(R^{-1}x_0)) \asymp C_{N,\gamma}$ and $h_\sigma\left(4, 2, R^{-1}x_0\right) \asymp C_{\gamma,\beta}$, where $C_{N,\gamma}, C_{\gamma,\beta}$ only depend on N, γ and γ, β respectively. Recall also that $R^\sigma \asymp R^2\mu_\gamma(B_R(x_0))/\mu_\beta(B_R(x_0))$.

Finally, we observe that when $\widetilde{H}_p$ is large, we have $\hat{\underline{\kappa}}(\widetilde{H}_p) \asymp \widetilde{H}_p^{-\frac{c_2\widetilde{H}_p^{1/2}}{m(1-m)}}$, where c_2 only depends on N, m, γ, β. Undoing the rescaling we obtain the lower bound (2.0.2) and the proof of Theorem 2.0.1 is concluded. □

Proof of Theorem 0.0.4. Once the positivity result is proven for solutions to the MDP, namely once Theorem 2.0.1 is established, then by a standard comparison argument, the positivity result can be extended to any nonnegative local (super)solution. For strong (super) solutions the result is immediate, while for more general concepts of solutions, such as weak energy or very weak, the proof follows by a long but straightforward limiting process; see [99] and [4] for more details about the non-weighted case; the case with weights follows along similar lines. □

Chapter 3

Harnack inequalities and Hölder continuity

In this third Chapter we study regularity estimates for nonnegative solutions to both linear and nonlinear equations.

3.1 The linear case

We are going to prove Harnack inequalities and local space-time Hölder continuity for nonnegative local solutions to the linear equation with Caffarelli-Kohn-Nirenberg weights. The equation

$$v_t = w_\gamma \sum_{i,j=1}^{N} \partial_i \left(A_{i,j}(t,x)\, \partial_j v \right), \tag{3.1.1}$$

is posed on the cylinder $Q := (0,T) \times \Omega$, where $A_{i,j} = A_{j,i}$ and for some $\gamma, \beta < N$ satisfying (0.0.11), i.e. $\gamma - 2 < \beta \le \left(\frac{N-2}{N}\right)\gamma$, as well we suppose that there exist constants $0 < \lambda_0 < \lambda_1 < +\infty$ such that

$$w_\gamma \asymp |x|^\gamma \qquad \text{and} \qquad 0 < \lambda_0 |x|^{-\beta} |\xi|^2 \le \sum_{i,j=1}^{N} A_{i,j}(t,x)\xi_i \xi_j \le \lambda_1 |x|^{-\beta} |\xi|^2 \,. \tag{3.1.2}$$

The regularity estimates that we present in this section are not present in the literature in the full range of parameters that we consider here, but several results have been obtained in different settings, see [62, 63, 64, 65, 89, 72, 71, 73, 74, 90]. We will only sketch the proofs, since they are minor modifications of those obtained by Chiarenza-Serapioni and Gutierrez-Wheeden, [63, 72, 71] combined with the original proof of Moser [74]. We shall keep track of the dependence of the Harnack constant by λ_0, λ_1 in a quantitative way as in [74], since in the nonlinear case this will have remarkable consequences.

In this weighted setting the Harnack inequality holds on suitable cylinders which take into account the geometry of the problem; recall that under assumptions (1), (2) or (3) we have

$$\rho_{x_0}^{\gamma,\beta}(R) := \left(\int_{B_R(x_0)} |x|^{(\beta-\gamma)\frac{N}{2}} \,\mathrm{d}x \right)^{\frac{2}{N}} \asymp \frac{\mu_\gamma(B_R(x_0))}{\mu_\beta(B_R(x_0))} R^2 \asymp R^{2+\beta-\gamma}.$$

The following cylinders generalize the standard parabolic ones:

$$\begin{aligned} Q_R(t_0,x_0) &:= \left\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d :\, t_0-\rho^{\gamma,\beta}_{x_0}(R)<t\le t_0\,,\ |x-x_0|<2\,R\right\}\,,\\ Q^+_R(t_0,x_0) &:= \left\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d :\, t_0-\frac14\,\rho^{\gamma,\beta}_{x_0}(R)<t\le t_0\,,\ |x-x_0|<\frac12\,R\right\}\,,\\ Q^-_R(t_0,x_0) &:= \left\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d :\, t_0-\frac78\,\rho^{\gamma,\beta}_{x_0}(R)<t\le t_0-\frac58\,\rho^{\gamma,\beta}_{x_0}(R)\,,\ |x-x_0|<\frac12\,R\right\}\,. \end{aligned} \tag{3.1.3}$$

It is convenient to introduce a suitable parabolic quasi-metric which carries on the information of the weights. Let $(t,x),(s,y)\in(0,\infty)\times\mathbb{R}^N$ we define

$$d_{\gamma,\beta}\left((t,x),(s,y)\right) := |x-y|\vee\left(\rho^{\gamma,\beta}_{\overline{xy}}\right)^{-1}(|t-s|), \tag{3.1.4}$$

where $\overline{xy}:=(x+y)/2$; the behaviour of $\left(\rho^{\gamma,\beta}_{\overline{xy}}\right)^{-1}$ is analyzed in Lemma 3.4.2. Similar quantities have already been introduced in [102] in order to prove Hölder continuity of the solutions to weighted parabolic equations similar to (3.1.2), but with different weights. In [65] it has been observed that although they imply continuity, they do not always imply Hölder continuity. For general classes of weights it is not possible to deduce any uniform modulus of continuity with respect to a standard parabolic quasi-distance. However, for our class of weights, we still manage to deduce Hölder continuity from Harnack inequalities. Indeed, the quasi-metric $d_{\gamma,\beta}$ is controlled (on bounded space-time domains) by the following, more standard, parabolic quasi-distance (see Lemma 3.4.2):

$$\tilde d_\sigma\left((t,x),(s,y)\right) := \begin{cases} |x-y|+|t-s|^{\frac1\sigma} & \text{if } \sigma=2+\beta-\gamma\ge 2,\\ |x-y|+|t-s|^{\frac12} & \text{if } 0<\sigma<2\,. \end{cases} \tag{3.1.5}$$

The following first result generalizes the Harnack inequality of Moser [74], in the spirit of Chiarenza-Serapioni [63, 64, 65] and Gutierrez-Wheeden [72, 71]:

Theorem 3.1.1 (Parabolic Harnack inequality in the linear case)**.** *Let v be a nonnegative bounded local weak solution to equation* (3.1.1) *on $Q:=(0,T)\times\Omega$, under assumption* (3.1.2)*. Then, for all $Q_R(t_0,x_0)\subset Q$, there exists $\overline{\kappa}_\ell>0$ such that*

$$\sup_{Q^-_R(t_0,x_0)} v \le \overline{\kappa}_\ell^{\,\lambda_0^{-1}+\lambda_1} \inf_{Q^+_R(t_0,x_0)} v. \tag{3.1.6}$$

The constant $\overline{\kappa}_\ell>0$ depends on N,γ,β, but not on v nor on λ_0,λ_1.

Remark. As remarked before, although this result has been proven before at least in some range of parameters, the dependence of the Harnack constant on the ellipticity constants λ_0,λ_1 was not clear nor explicit; such dependence is needed in the proof of Hölder continuity for nonlinear equations, as we will show at the end of this section; this was pointed out by Moser in [74], where a complete proof of (3.1.6) in the unweighted case $\beta=\gamma=0$ can be found. The (nontrivial) fact that $\overline{\kappa}_\ell$ only depends on N,γ,β is also pointed out by Gutierrez and Wheeden in [71] after the statement of their Harnack inequalities, Theorem A; indeed we sketch here an adaptation of their proof to our case.

As it often happens for linear parabolic equations, Hölder continuity follows by Harnack inequalities using a nowadays standard argument, cf. [73], that we sketch in the proof of Proposition 3.1.2. We will assume in what follows, without loss of generality, that $\overline{\kappa}_\ell \geq 2$ and $0 < \lambda_0 \leq \lambda_1$, and we define

$$\alpha := \log_A \frac{\overline{\kappa}_\ell^{\lambda_0^{-1}+\lambda_1}}{\overline{\kappa}_\ell^{\lambda_0^{-1}+\lambda_1} - 1} \in (0,1)\,, \tag{3.1.7}$$

where $A > 4$ which depends on γ, β, N and is given in (3.4.5). As well we introduce the notion of distance between sets of the form $Q = (0,T) \times \Omega \subset (0,\infty) \times \mathbb{R}^N$. Let $Q' = (T_1, T_2) \times \Omega' \subset Q$, we define

$$d_{\gamma,\beta}(Q,Q') := \inf_{\substack{(t,x)\in\{[0,T]\times\partial\Omega\}\cup\{\{0\}\times\Omega\},\\ (s,y)\in Q'}} |x-y| \vee \left(\rho_y^{\gamma,\beta}\right)^{-1}(|t-s|)\,. \tag{3.1.8}$$

We observe that if the $d_{\gamma,\beta}(Q,Q') = 2D$ then for any $(t,x) \in Q'$ the parabolic cylinder $Q_D(t,x) \subset Q$.

Proposition 3.1.2 (Hölder Continuity in the linear case)**.** *Let v be a nonnegative bounded local weak solution to equation* (3.1.1) *on $Q := (0,T) \times \Omega$, under the assumption* (3.1.2). *Let $Q' := (T_1,T_2) \times \Omega' \subset Q$ and let $2D = d_{\gamma,\beta}(Q,Q')$. Then there exist $\alpha \in (0,1)$ as in* (3.1.7) *and $\overline{\kappa}_\alpha > 0$, such that for all $(t,x),(s,y) \in Q'$*

$$\sup_{(t,x),(\tau,y)\in Q'} \frac{|v(t,x) - v(\tau,y)|}{d_{\gamma,\beta}\left((t,x),(s,y)\right)^\alpha} \leq \frac{\overline{\kappa}_\alpha}{D^\alpha} \|v\|_{\mathrm{L}^\infty(Q)}, \tag{3.1.9}$$

where $\overline{\kappa}_\alpha > 0$ depends only on $N, \gamma, \beta, \lambda_0, \lambda_1$.

The following corollary is immediate, and shows how the above estimates imply a more uniform modulus of continuity.

Corollary 3.1.3. *Under the assumptions of Proposition 3.1.2, there exist $\alpha \in (0,1)$ as in* (3.1.7) *and $\overline{\kappa}'_\alpha > 0$, such that for all $(t,x),(s,y) \in Q'$*

$$\sup_{(t,x),(\tau,y)\in Q'} \frac{|v(t,x) - v(\tau,y)|}{(|x-y| + |t-s|^{\frac{1}{2\vee\sigma}})^\alpha} \leq \frac{\overline{\kappa}'_\alpha}{D^\alpha} \|v\|_{\mathrm{L}^\infty(Q)}, \tag{3.1.10}$$

where $\overline{\kappa}'_\alpha > 0$ is given by

$$\overline{\kappa}'_\alpha = \overline{\kappa}_\alpha \overline{\kappa}_{19}^\alpha \begin{cases} 1\,, & \text{if } \sigma \geq 2, \\ \left(T^{\frac{1}{\sigma}} \vee \sup\limits_{x_0\in\Omega} |x_0|\right)^{\frac{\gamma-\beta}{2}}\,, & \text{if } 0 < \sigma < 2\,, \end{cases}$$

where $\overline{\kappa}_{19} > 0$ depends only on N, γ, β and is given in (3.4.2).

The proof of the above results relies on the following upper and lower bounds.

Proposition 3.1.4. *Let $u \in \mathrm{L}^p_{\gamma,\mathrm{loc}}((0,T) \times B_R(x_0))$ with $p > 0$ be a nonnegative local strong (sub)solution to* (3.1.1) *and let $x_0 \in \mathbb{R}^N$, $0 < R_1 < R_0 < R$ such that $0 \notin \overline{B_{R_0}(x_0) \setminus B_{R_1}(x_0)}$ and let $0 \leq T_0 < T_1 < T$. Then there exists a constant $\overline{\kappa}_{lin} > 0$ depending only on γ, β, N, p such that the following inequality holds*

$$\begin{aligned} \sup_{(\tau,y)\in(T_1,T]\times B_{R_1}(x_0)} u(\tau,y) &\leq \overline{\kappa}_{lin} \left[\frac{h_\sigma(R_0,R_1,x_0)}{(R_0-R_1)^\sigma} + \frac{1}{T_1 - T_0}\right]^{\frac{N-\gamma+\sigma}{\sigma p}} \\ &\quad \times \left[\int_{T_0}^T \int_{B_{R_0}(x_0)} u^p \frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right]^{\frac{1}{p}} \end{aligned} \tag{3.1.11}$$

where σ is defined in (0.0.21)*,* $h_\sigma(R_0, R_1, x_0)$ *is defined in* (1.1.1) *and* $\overline{\kappa}_{lin}$ *is a computable constant such that* $\overline{\kappa}_{lin} \lesssim S_{\gamma,\beta}^{\frac{2(N-\gamma)}{p\sigma}} \left(\lambda_0^{-1}\lambda_1\right)^{\frac{(N-\gamma+\sigma)}{\sigma p}}$*, with* $S_{\gamma,\beta}$ *as in Proposition 0.0.10.*

Proposition 3.1.5. *Let u be a nonnegative local strong (super)solution to* (3.1.1) *on* $(0,T)\times B_R(x_0)$*, with* $0 < R_1 < R_0 < R$ *such that* $0 \notin \overline{B_{R_0}(x_0) \setminus B_{R_1}(x_0)}$ *and let* $0 \le T_0 < T_1 < T$*. Then for any* $p > 0$ *there exists a constant* $\underline{\kappa}_{lin} > 0$ *depending only on* γ, β, N, p *such that the following inequality holds*

$$\begin{aligned} \inf_{(\tau,y)\in(T_1,T]\times B_{R_1}(x_0)} u(\tau,y) \ge \underline{\kappa}_{lin} &\left[\frac{h_\sigma\left(R_0,R_1,x_0\right)}{\left(R_0-R_1\right)^\sigma} + \frac{1}{T_1-T_0}\right]^{-\frac{N-\gamma+\sigma}{\sigma p}} \\ &\times \left[\int_{T_0}^{T}\int_{B_{R_0}(x_0)} u^{-p}\,\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right]^{-\frac{1}{p}} \end{aligned} \tag{3.1.12}$$

where σ is defined in (0.0.21)*,* h_σ *is defined in* (1.1.1) *and* $\underline{\kappa}_{lin}$ *is a computable constant such that* $\underline{\kappa}_{lin} \gtrsim S_{\gamma,\beta}^{-\frac{2(N-\gamma)}{p\sigma}} \left(\lambda_0^{-1}\lambda_1\right)^{-\frac{(N-\gamma+\sigma)}{\sigma p}}$ *where* $S_{\gamma,\beta}$ *is as in Proposition 0.0.10.*

Remark. The above estimates have been previously obtained by several authors in different settings, we just mention here the closest results: Lemma 3.17 of [71] (in the case of general $\mathcal{A}_2$ weights), Lemma 2.1 of [63], and Lemma 1 of [74] when there are no weights. The proof follows Moser's idea: using weighted Sobolev inequalities and upper (resp. lower) iterations , to obtain upper (resp. lower) space-time smoothing effects; indeed, the space-time upper bounds (3.1.11) can be obtained also by taking $m = 1$ in the proof of Theorem 1.3.1: note that the two factors u^{p+m-1} and u^p in the energy estimates (1.1.2) are now the same, hence the proof can be done directly with u, and we do not need to use the subsolution $v = u \vee 1$; as a consequence, the factor $+1$ in the integral in the right-hand side of formula (1.3.1) disappears. Analogously, the space-time lower bounds (3.1.12) follow by a minor modification of the proof of Proposition 2.4.1 with $m = 1$, more precisely we just repeat the Steps 1, 2 and 3 of the proof and we obtain the analogous of formula (2.4.11), which can be rewritten in the form (3.1.12) . Note that these proofs are considerably simpler than in the nonlinear setting, $m \in (0, 1)$.

Proof of Harnack inequalities, Theorem 3.1.1. The proof follows the lines of the original Moser proof in [74]. Once obtained local upper and lower bounds, (3.1.11) and (3.1.12), the hardest part of the proof consists in obtaining a reverse Hölder inequality that allows one to join them and deduce the Harnack inequalities (3.1.6). To our knowledge only two techniques are known to perform this task: one originally due to Moser [73] that gives a suitable reverse space-time Hölder's inequality on shifted cylinders, and another due to Bombieri and Giusti [101], see also [74], which shows how estimates (3.1.11) and (3.1.12) imply (local) absolute upper and lower bounds that allow to obtain the Harnack inequality (3.1.6). We will follow the latter strategy, and just sketch the proof, which is essentially the same as Section 4 and 5 of [71], see also [74]; we shall focus on the points where some non-straightforward changes are needed. Last, we just remark that it is enough to work in a cube of size 1, then the result will follow by rescaling.

• Step 1. *Bombieri-Giusti Lemma.* We are going to use a weighed version of Bombieri-Giusti Lemma as it has been done in Section 5 of [71] (see also Lemma 3 of [74]). Note that the following Lemma applies to measurable functions f, not necessarily solutions to a PDE.

Claim. Let $A, B, \overline{p}, \varrho, \delta$ be positive constants, and $Q_1, Q_{R_0}, Q_{R_1}, Q_\varrho$ as in (3.1.3). Also, we assume that the positive measurable function f defined on Q_1, and the doubling measure ν on $\mathbb{R}^{N+1}$ satisfy the inequalities

$$\begin{aligned} \sup_{Q_{R_1}} f^p &\le \frac{A}{(R_0-R_1)^B}\iint_{Q_{R_0}} f^p\nu(t,x)\,\mathrm{d}x\,\mathrm{d}t \qquad \text{and} \\ \nu\left\{(t,x)\in Q_1\,:\,\log f > s\right\} &\le \left(\frac{1}{s\,\overline{p}}\right)^{\delta}\nu(Q_1) \end{aligned} \tag{3.1.13}$$

for all $s>0$, $\frac{1}{2}\le\varrho\le R_1<R_0<1$, all $p\in(0,\overline{p})$. Then there exists $c_0=c_0(A,B,\delta)>0$ such that

$$\log\sup_{Q_\varrho} f \le \frac{c_0}{\overline{p}(1-\varrho)^{2B}}. \tag{3.1.14}$$

The proof of the above claim is a minor modification of the proof of Lemma 5.1 of [71], see Section 5 of [71] for more details. Indeed, for some range of parameters, for instance when our weights fall in the Muckenhoupt class $\mathcal{A}_2$, the proof is exactly the same. The only point where we can not directly adapt those proofs, is when a suitable "localized" weighted Poincaré inequality is used: in our context, such inequality reads

$$\int_{B_R(y_0)}|f(x)-\overline{f}|^2\frac{\varphi(x)}{|x|^\gamma}\,\mathrm{d}x \le c_\varphi\frac{\mu_\gamma(B_R(y_0))}{\mu_\beta(B_R(y_0))}R^2\int_{B_R(y_0)}|\nabla f|^2\frac{\varphi(x)}{|x|^\beta}\,\mathrm{d}x\,, \tag{3.1.15}$$

where $\overline{f}=\left(\int_{B_R(y_0)}\frac{\varphi(x)}{|x|^\gamma}\,\mathrm{d}x\right)^{-1}\left(\int_{B_R(y_0)}f(x)\frac{\varphi(x)}{|x|^\gamma}\,\mathrm{d}x\right)$, for any ball $B_R(y_0)\subset\mathbb{R}^N$ and for an extra "weight" $\varphi\in C_0(B_R(y_0))$, $0\le\varphi\le1$ with convex super-level sets, where

$$c_\varphi = c_{N,\gamma,\beta}\left(|B_R(y_0)|/\int_{B_R(y_0)}\varphi\,\mathrm{d}x\right)^2.$$

This inequality is proven in Lemma 4.1 of both [71, 72], and the relies on results of [103] (in the non-weighted case see Lemma 3 of [73]); this is the point where the restriction on the class of weights appears. We recall that the results of [71, 72] do not cover all the range of parameters γ,β that we consider here: they hold for weights which satisfy the $\mathcal{A}_2$ property (or generalizations of it), and this is not always the case in our setting. A closer inspection of the proof of Lemma 3 of [73] reveals that it is enough to prove (3.1.15) just for one function φ with the properties that $0\le\varphi\le1$ on $B_R(y_0)$, and for some $\delta\in(0,1)$ and some $R_\delta\in(0,R)$ we also have that $\varphi\ge\delta$ on $B_{R_\delta}(y_0)\subset B_R(y_0)$ and $\varphi=0$ on $\partial B_R(y_0)$. We are going to show that inequality (3.1.15) is indeed a consequence of the so-called Intrinsic Poincaré inequality

$$(\lambda_2-\lambda_1)\int_{B_R(y_0)}|f(x)-\overline{f}|^2\frac{\varphi_1^2(x)}{|x|^\gamma}\,\mathrm{d}x \le C_{\gamma,\beta}\int_{B_R(y_0)}|\nabla f|^2\frac{\varphi_1^2(x)}{|x|^\beta}\,\mathrm{d}x, \tag{3.1.16}$$

where φ_1 is the first eigenfunction of the operator $\mathcal{L}_{\gamma,\beta}$ (with Dirichlet boundary conditions and with unitary L^2_γ norm) on $B_R(y_0)$, $\overline{f}=\left(\int_{B_R(y_0)}f(x)\frac{\varphi_1^2(x)}{|x|^\gamma}\,\mathrm{d}x\right)$, and λ_1,λ_2 are respectively the first and the second eigenvalue of the $\mathcal{L}_{\gamma,\beta}$ on $B_R(y_0)$. The proof of inequality (3.1.16) is quite standard: this inequality is indeed equivalent to the second Poincaré inequality

$$\lambda_2\|g\|^2_{\mathrm{L}^2_\gamma(B)}\le\|\nabla g\|^2_{\mathrm{L}^2_\beta(B)} \qquad \text{for all } g\in\mathcal{D}_{\gamma,\beta}(B) \text{ such that} \qquad \int_\Omega g\varphi_1\frac{\mathrm{d}x}{|x|^\gamma}=0\,.$$

The above inequality is true on balls as a consequence of the compactness of the embedding $\mathcal{D}_{\gamma,\beta}(B)\subset\mathrm{L}^2_\gamma(B)$, where $\mathcal{D}_{\gamma,\beta}$ is defined in Subsection 0.0.1; finally, inequality (3.1.16) follows by letting $g = (u - \overline{u})\varphi_1$, see for instance Lemma 3.1 of [104]. The claim is proven.

• STEP 2. *Proof of the Harnack inequality.* We are going to apply twice the result of the previous step to get local upper and lower bounds that will finally combine in the Harnack estimates (3.1.6). In this case the proof is an adaptation of Section 6 of [71], see also Section 3 of [74] for the non-weighted case, we will just emphasize the essential changes.

Let us assume first that we are in the position to apply the Bombieri-Giusti inequality with $R_0 = 3/4$ and $R_1 = R = 2/3$ and $\varrho = 1/2$, to both $f \sim v$ and $f \sim v^{-1}$ on $Q_R^-(t_0, x_0)$ and $Q_R^+(t_0, x_0)$ respectively; we will briefly explain at the end of this step how to proceed to ensure that the assumptions (3.1.13) are satisfied by both v and v^{-1}. Using inequality (3.1.14) for $f = \mathrm{e}^{-M_2+V} v$ on $Q_R^-(t_0, x_0)$, where V and M_2 are chosen as in (3.1.19), we obtain the desired absolute local upper bounds:

$$\sup_{Q_{1/2}^-(t_0,x_0)} v = \mathrm{e}^{M_2-V} \sup_{Q_{1/2}^-(t_0,x_0)} f \le \mathrm{e}^{M_2-V} \exp\left(\frac{c_0}{\overline{p}(1/2)^{2B}}\right) \le \overline{\kappa}_{\ell,1}^{\lambda_0^{-1}+\lambda_1} \mathrm{e}^{-V}. \tag{3.1.17}$$

Proceeding analogously for $f = \mathrm{e}^{-M_2-V} v^{-1}$ on $Q_R^+(t_0, x_0)$, we obtain the desired absolute local lower bounds:

$$\inf_{Q_{1/2}^+(t_0,x_0)} v = \mathrm{e}^{-M_2-V} \left(\sup_{Q_{1/2}^+(t_0,x_0)} f\right)^{-1} \ge \mathrm{e}^{-M_2-V} \exp\left(-\frac{c_0}{\overline{p}(1/2)^{2B}}\right) \ge \overline{\kappa}_{\ell,2}^{-\lambda_0^{-1}-\lambda_1} \mathrm{e}^{-V}. \tag{3.1.18}$$

Note that the last inequalities in (3.1.17) and (3.1.18) follow by the choice of V, M_2 as in (3.1.19) and can be proven by following exactly Section 6 of [71], hence we omit the details. Finally, the Harnack inequality (3.1.6) follows by combining inequalities (3.1.17) and (3.1.18) and $\overline{\kappa}_\ell = \overline{\kappa}_{\ell,1} \cdot \overline{\kappa}_{\ell,2}$.

It only remains to show that we can actually use inequality (3.1.14) for $f = u$ and $f = u^{-1}$, hence we need to ensure the validity of hypotheses (3.1.13) in both cases. This is done by proving the so-called logarithmic estimates, see for instance Lemma 4.9 of [71]. The proof of that Lemma can be repeated also in our setting, and shows that: for any nonnegative bounded solution u defined on $(a, b) \times B_{3/2}$, bounded below by a positive constant in $(a, b) \times B_1$, then there exist c_1, M_2, δ and V such that, for any $s > 0$

$$\begin{aligned} \mu_\gamma \left\{(t, x) \in (t_0, b) \times B_1(x_0) \,:\, \log u < -s - M_2(b - t_0) - V\right\} &\le c_1 \left[\frac{1}{s} \frac{\mu_\gamma(B_1(x_0))}{\mu_\beta(B_1(x_0))} \frac{1}{b - t_0}\right]^\delta (b - t_0), \\ \mu_\gamma \left\{(t, x) \in (a, t_0) \times B_1(x_0) \,:\, \log u > s - M_2(a - t_0) - V\right\} &\le c_1 \left[\frac{1}{s} \frac{\mu_\gamma(B_1(x_0))}{\mu_\beta(B_1(x_0))} \frac{1}{t_0 - a}\right]^\delta (t_0 - a), \end{aligned} \tag{3.1.19}$$

where the constants $c_1, \delta > 0$ only depend on N, β, γ, $M_2 \sim \mu_\beta(B_1(x_0))/\mu_\gamma(B_1(x_0))$, and V depends on v, but it is the same in both cases, as explained carefully in Section 6 of of [71]. Details about the proof of the above estimates can be found in Section 4 of of [71], which in turn extend ideas of Moser (Section 2 and 3 of [74]) to the weighted case. The latter estimates, together with the local upper and lower bounds, (3.1.11) and (3.1.12), allow to apply the Bombieri-Giusti result in both cases. Hence (3.1.17) and (3.1.18) hold and the proof is concluded. The general statement follows by a scaling argument.

Finally we recall that, as Moser first noticed in [74], with the present method it is possible to keep track of the dependence on the "ellipticity" constants $0 < \lambda_0 \le \lambda_1 < +\infty$ throughout the proof: also in the present weighted setting we were able to keep track of such dependence in the constants. □

Proof of Hölder continuity, Proposition 3.1.2. We adapt the Moser's proof to our weighted case, see [73, 74]. We first prove how the oscillation of the solution decrease geometrically on parabolic cylinders. We recall that if the $d_{\gamma,\beta}(Q,Q')=2D$ then for any $(t,x)\in Q'$ the following inclusion holds $Q_D(t,x)\subset Q$. Fix $r\in(0,D/2)$ and denote for simplicity $Q_r:=Q_r(t_0,x_0)$ and $Q_r^{\pm}:=Q_r^{\pm}(t_0,x_0)$. Let us introduce the following quantities:

$$\overline{V}_r:=\sup_{Q_r} v\,,\quad \overline{V}_r^{\pm}:=\sup_{Q_r^{\pm}} v\,,\quad \underline{V}_r:=\inf_{Q_r} v\,,\quad \underline{V}_r^{\pm}:=\inf_{Q_r^{\pm}} v\,.$$

We are in the position to apply the Harnack inequality (3.1.6) to the nonnegative solution $\overline{V}_{2r}-v$ to obtain

$$\overline{V}_{2r}-\underline{V}_r^{-}=\sup_{Q_r^{-}}(\overline{V}_{2r}-v)\le H\inf_{Q_r^{+}}(\overline{V}_{2r}-v)=H\,(\overline{V}_{2r}-\overline{V}_r^{+})\,.$$

Note that without loss of generality we can set $H:=\overline{\kappa}_\ell^{\lambda_0^{-1}+\lambda_1}\ge 4$, with $\overline{\kappa}_\ell$ as in (3.1.6). Similarly, using $v-\underline{V}_{2r}$ we obtain $\overline{V}_r^{-}-\underline{V}_{2r}\le H\,(\underline{V}_r^{+}-\underline{V}_{2r})$ which, summed up with the previous inequality, gives

$$H\,(\overline{V}_r^{+}-\underline{V}_r^{+})+\overline{V}_r^{-}-\underline{V}_r^{-}\le(H-1)\,(\overline{V}_{2r}-\underline{V}_{2r})\,.$$

Using $Q_{r/A}\subset Q_r^{+}$ (see Lemma 3.4.3, formula 3.4.6), we conclude that

$$\operatorname*{osc}_{Q_{r/A}} v\le\operatorname*{osc}_{Q_r^{+}} v=\overline{V}_r^{+}-\underline{V}_r^{+}\le\frac{H-1}{H}\,(\overline{V}_{2r}-\underline{V}_{2r})=\frac{H-1}{H}\operatorname*{osc}_{Q_{2r}} v\,.$$

Recall that without loss of generality we have assumed that $H/(H-1)\le 4<A$, see also (3.1.7); a well-known iteration technique (see, *e.g.*, [96, Lemma 6.1]) then shows that

$$\operatorname*{osc}_{Q_r} v\le A^\alpha\frac{r^\alpha}{D^\alpha}\operatorname*{osc}_{Q_R} v\qquad\text{for all } r\in(0,D]\,,\tag{3.1.20}$$

with $\alpha:=\log(H/(H-1))/\log A\in(0,1)$, as in (3.1.7), and $H>1$ as above.

Now, we fix $(t,x),(s,y)\in Q'$, and we first assume that Ω' is convex. The first case that we analyze corresponds to $d_{\gamma,\beta}((t,x),(s,y))\le D$. Hence, there exists an integer $k\ge1$ such that

$$\frac{D}{A^{k+1}}\le d_{\gamma,\beta}((t,x),(s,y))<\frac{D}{A^k},$$

from which follows that $(t,x),(s,y)\in Q_{\frac{D}{A^k}}(t\vee s,\overline{xy})\subseteq Q_D(t\vee s,\overline{xy})\subset Q$, where $\overline{xy}=(x+y)/2$. Using 3.1.20 we get the following estimate

$$|v(t,s)-v(s,y)|\le\operatorname*{osc}_{Q_{D/A^k(\overline{xy},t\vee s)}} v\le\left(\frac{A\,D}{A^kD}\right)^\alpha\|v\|_{\mathrm{L}^\infty(Q)}\le\frac{A^{2\alpha}}{D^\alpha}d_{\gamma,\beta}((t,x),(s,y))^\alpha\,\|v\|_{\mathrm{L}^\infty(Q)}\,.\tag{3.1.21}$$

The second case corresponds to $d_{\gamma,\beta}((t,x),(s,y))>D$, and we proceed as follows

$$|v(t,s)-v(s,y)|\le2\|v\|_{\mathrm{L}^\infty(Q)}\le\frac{2\|v\|_{\mathrm{L}^\infty(Q)}}{D^\alpha}d_{\gamma,\beta}((t,x),(s,y))^\alpha\,.$$

The constant $\overline{\kappa}_\alpha>0$ is given by

$$\overline{\kappa}_\alpha:=1\vee A^\alpha\,,\tag{3.1.22}$$

where α is as in (3.1.7) and A as in (3.4.3); it depends on N, γ, β and λ_0, λ_1. In the case when Ω' is not convex, the result follows by a standard covering argument, however for the purposes of the present work, we only need quantitative information on balls. The proof is now concluded. □

Proof of Corollary 3.1.3. As a consequence of inequality (3.4.3) of Lemma 3.4.2, we know that there exist a constants $\overline{\kappa}' > 0$ such that for any $(t,x),(s,y) \in Q$ we have

$$d_{\gamma,\beta}((t,x),(s,y)) \le \overline{\kappa}' \left(|x-y| + |t-s|^{\frac{1}{2\vee\sigma}} \right),$$

which proves the Corollary. □

3.2 The nonlinear case

This Subsection essentially contains the proofs of Harnack inequalities and Hölder continuity for WFDE, Theorems 0.0.6 and 0.0.8 respectively.

Theorem 3.2.1 (Alternative form of Harnack inequality)**.** *Under the assumptions of Theorem 0.0.6, for any $t_0 > 0$ there exist constants $\overline{\kappa}_8, \overline{\kappa}_9', \kappa_* > 0$ such that*

$$\sup_{x\in B_R(x_0)} u(t,x) \le \overline{\kappa}_8 \frac{\|u(t_0)\|^{p\sigma\vartheta_p}_{\mathrm{L}^p_\gamma(B_{2R}(x_0))}}{t_0^{(N-\gamma)\vartheta_p}} + \overline{\kappa}_9' \inf_{x\in B_R(x_0)} u(t\pm\theta, x)$$

for any

$$t, t\pm\theta \in (t_0, t_0 + t_*(t_0)) \cap (0,T), \qquad \text{and} \qquad t_*(t_0) = \kappa_* R^\sigma \mu_\gamma(B_R(x_0))^{m-1} \|u(t_0)\|^{1-m}_{\mathrm{L}^1_{B_R(x_0)}}.$$

The constants $\overline{\kappa}_8, \overline{\kappa}_9', \kappa_* > 0$ depend on N, m, γ, β; $\kappa_* > 0$ is given in the proof of Corollary 2.8.2, and $\overline{\kappa}_9' = \overline{\kappa}_9 \underline{\kappa}^{-1}$ where $\overline{\kappa}_8, \overline{\kappa}_9 > 0$ are as in (1.0.1)*, while $\underline{\kappa} > 0$ has an (almost) explicit expression given in* (2.8.10)*; note that $\underline{\kappa}, \overline{\kappa}_8$ and $\overline{\kappa}_9$ depend on R and x_0 and, when $0 < m \le m_c$, $\underline{\kappa}$ depends also on $H_p(u_0, x_0, R)$ defined in* (0.0.23)*.*

Proof. It follows immediately by combining inequalities (1.0.1) and (0.0.25). □

Proof of the Harnack inequalities of Theorem 0.0.6. Due to the time translation invariance of the equation it suffices to prove the result for $t_0 = 0$. Assume $t \in (\varepsilon t_*, t_*)$, for $\varepsilon \in (0,1)$ fixed. Recall that $R^\sigma \asymp R^2 \mu_\gamma(B_R(x_0)) \mu_\beta(B_R(x_0))^{-1}$. Using the upper bound (0.0.22), inequality (0.0.20) and formula (0.0.23) we get

$$\begin{aligned}
\sup_{x\in B_R(x_0)} u(t,x) &\le \overline{\kappa}_1 \frac{\|u_0\|^{p\sigma\vartheta_p}_{\mathrm{L}^p_\gamma(B_{2R}(x_0))}}{t^{(N-\gamma)\vartheta_p}} + \overline{\kappa}_2 \left[\frac{t}{\overline{\kappa}_{17} R^\sigma} \right]^{\frac{1}{1-m}} \\
&\le \left[\frac{\overline{\kappa}_1 \, \|u_0\|^{p\sigma\vartheta_p}_{\mathrm{L}^p_\gamma(B_{2R}(x_0))} R^{\frac{\sigma}{1-m}}}{(\varepsilon t_*)^{(N-\gamma)\vartheta_p + \frac{1}{1-m}}} + \frac{\overline{\kappa}_2}{\overline{\kappa}_{17}^{\frac{1}{1-m}}} \right] \left[\frac{t}{R^\sigma} \right]^{\frac{1}{1-m}} \\
&\le \left[\frac{\overline{\kappa}_1 \omega_\gamma^{\sigma\vartheta_p}}{2^{\frac{\sigma}{1-m}}} \frac{H_p(u_0, x_0, 2R)}{\varepsilon^{\frac{p\sigma\vartheta_p}{1-m}}} + \frac{\overline{\kappa}_2}{\overline{\kappa}_{17}^{\frac{1}{1-m}}} \right] \left[\frac{t}{R^\sigma} \right]^{\frac{1}{1-m}} \\
&\le \left[\frac{\overline{\kappa}_1 \omega_\gamma^{\sigma\vartheta_p}}{2^{\frac{\sigma}{1-m}}} + \frac{\overline{\kappa}_2}{\overline{\kappa}_{17}^{\frac{1}{1-m}}} \right] \left[1 \vee \frac{H_p(u_0, x_0, 2R)}{\varepsilon^{\frac{\sigma p\vartheta_p}{1-m}}} \right] \left[\frac{t}{R^\sigma} \right]^{\frac{1}{1-m}},
\end{aligned} \tag{3.2.1}$$

where $\omega_\gamma = B_1(0)$ and $\overline{\kappa}_{17}$ as in inequality (0.0.20). We recall next the lower bound (0.0.25) , that in this case reads

$$\inf_{x\in B_R(x_0)} u(t,x) \ge \inf_{x\in B_{2R}(x_0)} u(t,x) \ge \underline{\kappa}\left[\frac{t}{(2R)^\sigma}\right]^{\frac{1}{1-m}} \qquad \text{for any } t\in[0,t_*]\cap(0,T).$$

By combining the two above inequalities we get for any $t\in(\varepsilon t_*, t_*)$:

$$\sup_{x\in B_R(x_0)} u(t,x) \le \left[\frac{\overline{\kappa}_1\omega_\gamma^{\sigma\vartheta_p}}{\underline{\kappa}} + \frac{\overline{\kappa}_2 2^{\frac{\sigma}{1-m}}}{\underline{\kappa}\,\overline{\kappa}_{17}^{\frac{1}{1-m}}}\right]\left[1\vee\frac{H_p(u_0,x_0,2R)}{\varepsilon^{\frac{\sigma p\vartheta_p}{1-m}}}\right]\inf_{x\in B_R(x_0)} u(t,x) := \overline{\kappa}_3 \inf_{x\in B_R(x_0)} u(t,x)\,.$$

The constants $\overline{\kappa}_1,\overline{\kappa}_2>0$ depend on N,m,γ,β and are given in (0.0.22); $\underline{\kappa}>0$ is given in (2.8.10): notice that, when $0<m\le m_c$, $\underline{\kappa}$ depends also on $H_p(u_0,x_0,2R)$ defined in (0.0.23). We finally recall that

$$\begin{aligned}\overline{\kappa}_3 &:= \left[\frac{\overline{\kappa}_1\omega_\gamma^{\sigma\vartheta_p}}{\underline{\kappa}} + \frac{\overline{\kappa}_2 2^{\frac{\sigma}{1-m}}}{\underline{\kappa}\,\overline{\kappa}_{17}^{\frac{1}{1-m}}}\right]\left[1\vee\frac{H_p(u_0,x_0,2R)}{\varepsilon^{\frac{\sigma p\vartheta_p}{1-m}}}\right]\\ &\asymp \left[\overline{\kappa}_1\omega_\gamma^{\sigma\vartheta_p} + \frac{\overline{\kappa}_2 2^{\frac{\sigma}{1-m}}}{\overline{\kappa}_{17}^{\frac{1}{1-m}}}\right]\frac{H_p(u_0,x_0,2R)}{\varepsilon^{\frac{\sigma p\vartheta_p}{1-m}}}\left(\frac{\widetilde{H}_p^{c_2}}{R^{c_1}}\right)^{\frac{\widetilde{H}_p^{1/2}}{m(1-m)}} \qquad \text{when} \quad \widetilde{H}_p\gg 1\,,\end{aligned} \tag{3.2.2}$$

where $\widetilde{H}_p(u_0,x_0,2R) := 1+\left(\frac{|x_0|}{2R}\vee 1\right)^{\beta-\gamma} H_p(u_0,x_0,2R)^{1-m} \ge 1$ and $c_1,c_2>0$ only depend on N,m,p,β,γ. See also Corollary 2.8.2 for a more detailed the expression of $\underline{\kappa},c_1,c_2$. This concludes the proof. $\square$

We can prove an analogous continuity result for local solutions to the (WFDE), using the upper and lower bounds of Theorems 1.0.1 and 0.0.4, and the linear results of the previous subsection.

Proof of the Hölder continuity estimate of Theorem 0.0.8. We split the proof in two steps.
• STEP 1. *Intrinsic rescaling.* We begin by considering a local solution u on the cylinder $Q:=(0,T]\times\Omega$. Fix $t_0,R_0>0$ such that $Q^*_{4R_0}(t_0,x_0):=(t_0,T\wedge(t_0+t_*)]\times B_{4R_0}(x_0)\subset Q$. We define the rescaled solution $\hat{u}$ as follows:

$$\hat{u}(\hat{t},\hat{x}) := M_0^{-1}u(t,x) \qquad \text{with} \qquad t=R_0^\sigma M_0^{1-m}\hat{t}\,,\ x=R_0\hat{x},$$

where M_0 is any positive real number such that $M_0\ge\|u\|_{\mathrm{L}^\infty(Q_{4R_0}(t_0,x_0))}$. It is easy to check that if u is a local solution on $Q^*_{4R_0}(t_0,x_0)$, then $\hat{u}$ is a local solution to the same equation on $Q^*_4(\hat{t}_0,\hat{x}_0):=[\hat{t}_0,\hat{T}\wedge(\hat{t}_0+\hat{t}_*)]\times B_4(\hat{x}_0)$, where

$$\hat{t}_* = \hat{t}_*(\hat{u}(\hat{t}_0),\hat{x}_0,4) = \kappa_*\,4^\sigma\frac{\|\hat{u}(\hat{t}_0)\|^{1-m}_{\mathrm{L}^1_\gamma(B_4(\hat{x}_0))}}{\mu_\gamma(B_4(\hat{x}_0))^{1-m}} = \kappa_*\,4^\sigma\left[\frac{\|u(\hat{t}_0)\|_{\mathrm{L}^1_\gamma(B_4(\hat{x}_0))}}{M_0\,\mu_\gamma(B_4(\hat{x}_0))}\right]^{1-m}.$$

Moreover, $\|\hat{u}\|_{\mathrm{L}^\infty(Q^*_4(\hat{t}_0,\hat{x}_0))}\le 1$, since by assumption we have $M_0\ge\|u\|_{\mathrm{L}^\infty(Q_{4R_0}(t_0,x_0))}$.
We are now in the position to apply the lower bounds of Theorem 0.0.4 to $\hat{u}$ on $Q^*_4(\hat{t}_0,\hat{x}_0)$:

$$\inf_{x\in B_2(\hat{x}_0)}\hat{u}(t,x)\ge\underline{\kappa}\left[\frac{\hat{t}-\hat{t}_0}{2^\sigma}\right]^{\frac{1}{1-m}} \qquad \text{for any } \hat{t}\in[\hat{t}_0+\tfrac14\hat{t}_*,\hat{t}_0+\hat{t}_*]\cap(0,\hat{T}), \tag{3.2.3}$$

where 2 is the radius of the ball. Note that $\underline{\kappa}$ has an (almost) explicit expression is given in (2.8.10), and (in the very fast diffusion range, i.e. when $m < m_c$ and $p > 1$) depends on $H_p(\hat{u}(\hat{t}_0), \hat{x}_0, 4)$ defined in (0.0.23). Clearly inequality (3.2.3) implies

$$\inf_{(t,x)\in Q_2^*(\hat{t}_0,\hat{x}_0)} \hat{u}(t,x) = \inf_{(t,x)\in[\hat{t}_0+\hat{t}_*/2,\hat{t}_0+\hat{t}_*]\cap(0,\hat{T})\times B_2(\hat{x}_0)} \hat{u}(t,x) \ge 4^{-\frac{\sigma}{1-m}}\underline{\kappa}\,\hat{t}_*^{\frac{1}{1-m}}\,.$$

• Step 2. *Application of the linear result.* $\hat{u}$ can be considered a solution to the linear equation (3.1.1) with $a(t,x) = m\hat{u}^{m-1}(t,x)$; we are now in the position to apply the result of Corollary 3.1.3 inside the cylinder $Q_2^* := [\hat{t}_0 + \hat{t}_*/2, \hat{t}_0 + \hat{t}_*] \cap (0,\hat{T}) \times B_2(\hat{x}_0)$, since in Q_2^* we have

$$\lambda_0 := m \le m\hat{u}^{m-1} = a(t,x) \le 4^{\sigma}\underline{\kappa}^{m-1}\hat{t}_*^{-1} =: \lambda_1\,. \tag{3.2.4}$$

Then, on the cylinder $Q_1^* := [\hat{t}_0 + (5/8)\hat{t}_*, \hat{t}_0 + (7/8)\hat{t}_*] \cap (0,\hat{T}) \times B_1(\hat{x}_0)$, the result of Corollary 3.1.3 implies that there exist $\alpha \in (0,1)$ given in (3.1.7), and $\overline{\kappa}'_\alpha > 0$, depending on $N, \gamma, \beta, \lambda_0, \lambda_1$ such that

$$\sup_{(\hat{t},\hat{x}),(\hat{\tau},\hat{y})\in Q_1^*} \frac{|\hat{u}(\hat{t},\hat{x}) - \hat{u}(\hat{\tau},\hat{y})|}{\left(|\hat{x}-\hat{y}| + |\hat{t}-\hat{\tau}|^{\frac{1}{2\vee\sigma}}\right)^{\alpha}} \le \frac{\overline{\kappa}'_\alpha}{D^\alpha}\|\hat{u}\|_{\mathrm{L}^\infty(Q_{4\hat{R}})} \le \frac{\overline{\kappa}'_\alpha}{D^\alpha}\,, \tag{3.2.5}$$

where we have used that $\|\hat{u}\|_{\mathrm{L}^\infty(Q_{4\hat{R}})} \le \|\hat{u}\|_{\mathrm{L}^\infty(Q_4^*(\hat{t}_0,\hat{x}_0))} \le 1$, with $D = d_{\gamma,\beta}(Q_2^*, Q_1^*)$ defined in (3.1.8). Due to the particular form of the cylinders Q_1^* and Q_2^*, we have that

$$D = 1 \wedge \inf_{\hat{y}\in B_1(\hat{x})} \left(\rho_{\hat{y}}^{\gamma,\beta}\right)^{-1} (\hat{T}\wedge\hat{t}_*/8) \ge 1 \wedge \overline{\kappa}_{19}^{-2}(\hat{T}\wedge\hat{t}_*/8)^{1/\sigma} \wedge \overline{\kappa}_{19}^{-2}\left(\rho_{\hat{x}_0}^{\gamma,\beta}\right)^{-1}(\hat{T}\wedge\hat{t}_*/8) := D_0\,.$$

The latter inequality follows by assumptions (1), (2) or (3). Undoing the intrinsic change of variables, (3.2.5) transforms into (0.0.27) and the proof is concluded.

Finally, note that when H_p is large enough, by Corollary 2.8.2 we know that $\underline{\kappa} \sim \left(R^{c_4}/\widetilde{H}_p^{c_5}\right)^{\widetilde{H}_p^{1/2}/m(1-m)}$, hence $\alpha = \log_A \frac{\overline{\kappa}_\ell^{\lambda_0^{-1}+\lambda_1}}{\overline{\kappa}_\ell^{\lambda_0^{-1}+\lambda_1}-1}$ given in (3.1.7), with λ_0, λ_1 given in (3.2.4), behaves like $\alpha \sim \exp\left(-\frac{c_6}{t_*}\widetilde{H}_p^{\frac{c_7}{m}\widetilde{H}_p^{1/2}}\right)$, recalling that $c_i > 0$ only depend on N, m, p, β, γ. $\square$

Appendix

We collect in this Appendix several technical facts and proofs, used in the rest of Part I.

3.3 Appendix-A

This Appendix is devoted to the proof of the upper and lower energy estimates of Lemma 1.1.1 and of the Caccioppoli estimate of Lemma 1.1.2.

Approximation (via truncation) of powers of strong solutions. The proof of the energy estimates relies on the idea of using $u^{p-1}\psi$ as a test function, where ψ is a suitable smooth cutoff function and u is a solution to the WFDE. As the reader may guess, this is not an admissible test function, hence we need to proceed by a careful approximation. An additional difficulty is represented by the presence of singular/degenerate weights: under our assumptions, u is merely a function in $C_{\rm loc}((T_0,T);\mathrm{L}^p_{\gamma,{\rm loc}}(\Omega))$ such that $u^m\in \mathrm{L}^2_{\rm loc}((T_0,T);H^1_{\gamma,\beta,{\rm loc}}(\Omega))$; as already observed, u need not be a function in $\mathrm{L}^1_{\rm loc}(\Omega)$ and its gradient ∇u needs not to be the distributional one, see [57, 80]. The goal of the next Lemma is to show that a suitable truncation of a strong solution to WFDE belongs to the class of admissible test functions, hence an approximate energy identity holds. Here we follow the approach used in [98] and in [73]. Let $p>1$ and $1<l<k$: we define following auxiliary functions for $u>0$

$$J_p(u):=\begin{cases} ((u\wedge k)^{\frac{p-1}{m}}-l^{-1})_+ & \text{if } 1<p\le 1+m,\\ (u\wedge l)^{\frac{p-1}{m}-1}(u\wedge k) & \text{if } p>1+m,\end{cases}\qquad G_p(u):=\int_0^u J_p(s^m)\,\mathrm{d}s\,.$$

Note that J_p is a bounded Lipschitz function, for all choices of $k>l>1$ and $p>1$. Recall that u^{p-1} is not an admissible test function, hence we will use a truncation of it, in the precise form of $J_p(u^m)$.

Lemma 3.3.1. *Let u be a non-negative strong local solution to WFDE in $(T_0,T)\times B_R(x_0)$. For every $p>1$ and for any $[t_1,t_2]\subset(T_0,T)$ the following equality holds*

$$\int_{t_1}^{t_2}\int_{B_R(x_0)} u_t J_p(u^m)\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{t_2}\int_{B_R(x_0)}\nabla u^m\cdot\nabla(J_p(u^m)\psi)\,\mathrm{d}x\,\mathrm{d}t=0,\tag{3.3.1}$$

for any $\psi\in C^2((T_0,T);C^2_c(B_R(x_0)))$. A local strong sub (resp. super) solution satisfies (3.3.1) *with $\le$ (resp. $\ge$) for any nonnegative test function in the same class.*

Proof. By definition $u^m\in \mathrm{L}^2_{\rm loc}((T_0,T);H^1_{\gamma,\beta,{\rm loc}}(B_R(x_0)))$, hence there exists a sequence of functions $\phi_n\in C_c^\infty((T_0,T)\times B_R(x_0))$ which converges strongly to u^m in $\mathrm{L}^2_{\rm loc}(T_0,T;H^1_{\gamma,\beta,{\rm loc}}(B_R(x_0)))$. Since $J_p(\cdot)$ is a Lipschitz function, the family $\{\psi J_p(\phi)\}$ is a subset of $W^{1,2}_{\rm loc}(T_0,T;\mathrm{L}^2_\gamma(K))\cap \mathrm{L}^2_{\rm loc}(T_0,T;\mathcal{D}_{\gamma,\beta}(B_R(x_0)))$, hence an admissible test function in the sense of Definition 0.0.1, so that

$$\begin{aligned}\int_\Omega &[u(t_2,x)\ \psi(t_2,x)J_p(\phi_n)(t_2,x)-u(t_1,x)\psi(t_1,x)J_p(\phi_n)(t_1,x)]\,|x|^{-\gamma}\,\mathrm{d}x\\ &=\int_{t_1}^{t_2}\int_\Omega u(\psi J_p(\phi_n))_t\ |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t-\int_{t_1}^{t_2}\int_\Omega \nabla u^m\cdot\nabla(\psi J_p(\phi_n))\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Integrating by parts in time the first integral on the right-hand side, we obtain

$$\int_{t_1}^{t_2}\int_{B_R(x_0)} u_t J_p(\phi_n)\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{t_2}\int_{B_R(x_0)}\nabla u^m\cdot\nabla(J_p(\phi_n)\psi)\,\mathrm{d}x\,\mathrm{d}t=0; \tag{3.3.2}$$

the reader may observe that this integration by parts makes sense since u is assumed to be a strong solution, i.e. $u_t\in \mathrm{L}^1((T_0,T)\times B_R(x_0))$. Taking the limit as $n\to\infty$ in (3.3.2) gives (3.3.1). $\square$

Proof of the energy estimates of Lemma 1.1.1

We split the proof of Lemma 1.1.1 in several parts: first we prove the upper estimate, then the lower.

Proof of the upper energy inequality (1.1.2) Let us fix $x_0\in\mathbb{R}^N$, and simply denote $B_R=B_R(x_0)$ when there is no ambiguity.

• Step 1. *Reduction.* The upper energy inequality (1.1.2) follows by a slightly different inequality:

$$\begin{aligned}\int_{B_{R_1}} &u(T,x)^p\,|x|^{-\gamma}\,\mathrm{d}x+\int_{T_1}^{T}\int_{B_{R_1}}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t\\ &\le C\,(m,p)\left[\frac{h_\sigma\,(R,R_1,x_0)}{(R-R_1)^\sigma}+\frac{1}{T_1-T_0}\right]\int_{T_0}^{T}\int_{B_R}\left(u^{p+m-1}+u^p\right)\ |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{3.3.3}$$

Indeed, inequality (1.1.2) follows from (3.3.3) by letting $T=\tau$ and taking the supremum in $\tau\in[T_1,T]$.

• Step 2. *First energy inequality.* In this step we want to prove the following inequality:

$$\begin{aligned}\frac{p-1}{p}&\int_{B_R}\left[u(T,x)^p\ \psi^2\,(T,x)-u(T_0,x)^p\psi^2\,(T_0,x)\right]|x|^{-\gamma}\,\mathrm{d}x\\ &\quad+\frac{2m\,(p-1)^2}{(p+m-1)^2}\int_{T_0}^{T}\int_{B_R}\psi^2\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta}\\ &\quad\le 2\left[\int_{T_0}^{T}\int_{B_R}u^p\psi|\psi_t|\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}+m\int_{T_0}^{T}\int_{B_R}u^{p+m-1}|\nabla\psi|^2\ \frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta}\right].\end{aligned} \tag{3.3.4}$$

Following Moser's approach [73], we would like to test the equation with $u^{p-1}\psi^2$, but unfortunately this is not an admissible test function: we shall proceed by approximation, using $J_p(u^m)\psi^2$ as in Lemma 3.3.1; this approximation extends to our weighted setting some ideas from Aronson and Serrin [98]:

$$\int_{T_0}^{T}\int_{B_R}u_tJ_p(u^m)\psi^2|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t+\int_{T_0}^{T}\int_{B_R}\nabla u^m\cdot\nabla(J_p(u^m)\psi^2)|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t=0. \tag{3.3.5}$$

Recalling that $\partial_t G_p(u) = J_p(u^m)u_t$, an integration by parts (in time) in the left-hand side of (3.3.5) gives

$$\begin{aligned}\int_{B_R} &\left[G_p(u)(T,x)\,\psi^2\,(T,x) - G_p(u)(T_0,x)^p\psi^2\,(T_0,x)\right]\;|x|^{-\gamma}\,\mathrm{d}x \\ &+\int_{T_0}^{T}\int_{B_R}\nabla u^m\cdot\nabla(J_p(u^m)\psi^2)|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t \le 2\int_{T_0}^{T}\int_{B_R}|\psi||\psi_t|G_p(u)|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.3.6}$$

Note that $J_p(u^m)\to\tilde{J}_p(u^m)$ and $G_p(u)\to\tilde{G}_p(u)$ as $k\to\infty$ where

$$\tilde{J}_p(u) := \begin{cases} (u^{\frac{p-1}{m}} - l^{-1})_+ & \text{if } 1<p\le 1+m, \\ (u\wedge l)^{\frac{p-1}{m}-1}\,u & \text{if } p>1+m, \end{cases} \quad \tilde{G}_p(u) := \int_0^u \tilde{J}_p(s^m)\,\mathrm{d}s\,.$$

Since J_p is Lipschitz, taking limits as $k\to\infty$ in inequality (3.3.6) we get (by dominated convergence) :

$$\begin{aligned}\int\limits_{B_R}&\left[\tilde{G}_p(u)(T,x)\;\psi^2\,(T,x) - \tilde{G}_p(u)(T_0,x)^p\psi^2\,(T_0,x)\right]\frac{\mathrm{d}x}{|x|^\gamma} + \int\limits_{T_0}^{T}\int\limits_{B_R}\psi^2\tilde{J}_p{}'(u^m)\,|\nabla u^m|^2\,\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta} \\ &\le 2\int_{T_0}^{T}\int_{B_R}|\psi||\psi_t|\tilde{G}_p(u)\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma} - 2\int_{T_0}^{T}\int_{B_R}\psi\tilde{J}_p(u^m)\nabla u^m\cdot\nabla\psi\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta}.\end{aligned} \tag{3.3.7}$$

We combine now the following numerical inequality

$$\tilde{J}_p^2(u^m) \le \left(\frac{m}{p-1}\right)\tilde{J}_p'(u^m)u^{p+m-1}\left[1+\left(\frac{p-1}{m}-1\right)\mathbb{1}\{u^m>l\}\right]$$

with Young's inequality $|v\cdot w|\le |v|^2/4+|w|^2$ to obtain

$$2|\psi\tilde{J}_p(u^m)\nabla u^m\cdot\nabla\psi| \le \frac{1}{2}\psi^2\tilde{J}_p{}'(u^m)|\nabla u^m|^2 + \frac{2m}{p-1}u^{p+m-1}|\nabla\psi|^2 f(l,u), \tag{3.3.8}$$

where $f(l,u) = \left[1+\left(\frac{p-1}{m}-1\right)\mathbb{1}\{u^m>l\}\right]$. Combining (3.3.7) and (3.3.8) we get

$$\begin{aligned}\int_{B_R}&\left[\tilde{G}_p(u)(T,x)\;\psi^2\,(T,x) - \tilde{G}_p(u)(T_0,x)^p\psi^2\,(T_0,x)\right]\frac{\mathrm{d}x}{|x|^\gamma} + \frac{1}{2}\int_{T_0}^{T}\int_{B_R}\psi^2\tilde{J}_p{}'(u^m)\,|\nabla u^m|^2\,\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta} \\ &\le 2\int_{T_0}^{T}\int_{B_R}|\psi||\psi_t|\tilde{G}_p(u)\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma} + \frac{2m}{p-1}\int_{T_0}^{T}\int_{B_R}u^{p+m-1}|\nabla\psi|^2 f(l,u)\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\beta}.\end{aligned}$$

Finally, we obtain (3.3.4) by taking the limit as $l\to\infty$ in the above inequality: we notice that in such limit $\tilde{J}_p{}'(u^m)\,|\nabla u^m|^2 \to \tilde{c}_{p,m}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2$ in the appropriate integral sense, where $\tilde{c}_{p,m}$ is a suitable multiplicative constant, as well $\tilde{G}_p\to u^p/p$ and $f(l,u)\to 1$ by dominated convergence.

• STEP 3. *Choice of the test function* ψ. By a suitable choice of test function, we can show that inequality (3.3.4) implies (3.3.3). It is always possible to choose a smooth $0\le\psi\le 1$ supported in $[T_0,T]\times B_R$, such that $\psi\equiv 1$ on $[T_1,T]\times B_{R_1}$, $\psi(T_0,x)=0$ for all $x\in B_R$ and $\psi(t,x)=0$ for all $(t,x)\in[T_0,T]\times\partial B_R$, so that there exists $K_\psi>0$ (depending only on N) such that $|\nabla\psi(t,x)|^2\le K_\psi\,(R-R_1)^{-2}$ and $|\psi_t(t,x)|\le K_\psi(T_1-T_0)^{-1}$ for all $(t,x)\in(T_0,T]\times B_R\setminus B_{R_1}$. With this test function, we estimate the two sides of (3.3.4) separately.

Estimating the right-hand side of (3.3.4). We will show that

$$\left[\int_{T_0}^{T}\int_{B_R} u^p\ \psi\ |\psi_t|\ |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t+\int_{T_0}^{T}\int_{B_R} u^{p+m-1}\ |\nabla\psi|^2\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t\right]$$
$$\le 2K_\psi\left[\frac{1}{T_1-T_0}+\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma}\right]\left[\int_{T_0}^{T}\int_{B_R} u^p\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}+\int_{T_0}^{T}\int_{B_R} u^{p+m-1}\frac{\mathrm{d}x\,\mathrm{d}t}{|x|^\gamma}\right], \tag{3.3.9}$$

where the function $h_\sigma(R,R_1,x_0)$ is defined in (1.1.1). Indeed, plugging the above chosen ψ in the right-hand side of inequality (3.3.4) we get

$$c_{m,p}\left[\int_{B_R} u(T,\cdot)^p\ \psi^2(T,\cdot)\ |x|^{-\gamma}\,\mathrm{d}x+\int_{T_0}^{T}\int_{B_R}\psi^2\ |\nabla u^{\frac{p+m-1}{2}}|^2\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t\right]$$
$$\le 2\left[\int_{T_0}^{T}\int_{B_R} u^p\ \psi\ |\psi_t|\ |x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t+\int_{T_0}^{T}\int_{B_R} u^{p+m-1}\ |\nabla\psi|^2\ |x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t\right] \tag{3.3.10}$$

where the constant $c_{m,p}$ is given by

$$c_{m,p}=\frac{p-1}{p}\wedge\frac{2m\,(p-1)^2}{(p+m-1)^2}.$$

We just have to estimate the quotient $|x|^{-\beta}/|x|^{-\gamma}$ in the right-hand side of (3.3.10) in terms of $h_\sigma(R,R_1,x_0)$ and $R-R_1$ to get (3.3.9). First, recall that $\nabla\psi(t,\cdot)$ is supported in $B_R(x_0)\setminus B_{R_1}(x_0)$ for all $t\in(T_0,T_1]$. Next we split two cases, namely $\sigma<2$ and $\sigma\ge 2$.

– *Case* $0<\sigma<2$. In $\overline{B_R(x_0)\setminus B_{R_1}(x_0)}$ we have $|x|^{-\beta}=|x|^{\gamma-\beta}|x|^{-\gamma}\le(|x_0|+R)^{\gamma-\beta}\,|x|^{-\gamma}$, hence

$$\frac{(|x_0|+R)^{\gamma-\beta}}{(R-R_1)^2}=\left(\frac{|x_0|+R}{R-R_1}\right)^{\gamma-\beta}\frac{1}{(R-R_1)^\sigma}=\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma}, \tag{3.3.11}$$

since we recall that $0<\sigma=2+\beta-\gamma<2$ means $\gamma>\beta$ and $\frac{R+|x_0|}{R-R_1}\ge\frac{R}{R-R_1}>1$.

– *Case* $\sigma\ge 2$. Recall that $\sigma\ge 2$ means that $\gamma\le\beta$. We now consider two sub-cases. Recall that we consider balls $B_{R_1}(x_0)\subset B_R(x_0)$ such that $0\notin\overline{B_R(x_0)\setminus B_{R_1}(x_0)}$.

If $0\notin B_R(x_0)$, i.e. $|x_0|>R$, then in $\overline{B_R(x_0)\setminus B_{R_1}(x_0)}$ we have

$$|x|^{-\beta}=|x|^{-\gamma}/|x|^{\beta-\gamma}\le|x|^{-\gamma}/\left(|x_0|-R\right)^{\beta-\gamma},$$

hence

$$\frac{(|x_0|-R)^{\gamma-\beta}}{(R-R_1)^2}=\left(\frac{R-R_1}{|x_0|-R}\right)^{\beta-\gamma}\frac{1}{(R-R_1)^\sigma}\le\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma}. \tag{3.3.12}$$

If $0\in B_{R_1}(x_0)$, i.e. $R_1>|x_0|$ then in $\overline{B_R(x_0)\setminus B_{R_1}(x_0)}$ we have

$$|x|^{-\beta}=|x|^{-\gamma}/|x|^{\beta-\gamma}\le|x|^{-\gamma}/(R_1-|x_0|)^{\beta-\gamma},$$

hence

$$\frac{(R_1-|x_0|)^{\gamma-\beta}}{(R-R_1)^2}=\left(\frac{R-R_1}{R_1-|x_0|}\right)^{\beta-\gamma}\frac{1}{(R-R_1)^\sigma}\le\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma}. \tag{3.3.13}$$

Estimating the left-hand side and the right-hand side of (3.3.10). We just observe that since $\psi = 1$ in $[T_1, T] \times B_{R_1}$ we get

$$\int_{B_{R_1}} u(T,x)^p \ |x|^{-\gamma} \, \mathrm{d}x \le \int_{B_R} u(T,x)^p \ \psi^2 \left(T,x\right) \ |x|^{-\gamma} \, \mathrm{d}x,$$

and

$$\int_{T_1}^{T} \int_{B_{R_1}} |\nabla u^{\frac{p+m-1}{2}}|^2 \ |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t \le \int_{T_0}^{T} \int_{B_R} \psi^2 \ |\nabla u^{\frac{p+m-1}{2}}|^2 \ |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t.$$

Summing up, inequality (3.3.10) becomes

$$\begin{aligned} c_{m,p} & \left[\int_{B_{R_1}} u(T,x)^p |x|^{-\gamma} \, \mathrm{d}x + \int_{T_1}^{T} \int_{B_{R_1}} |\nabla u^{\frac{p+m-1}{2}}|^2 \ |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t \right] \\ & \le 2K_\psi \left[\frac{h_\sigma \left(R, R_1, x_0\right)}{\left(R-R_1\right)^\sigma} + \frac{1}{T_1 - T_0} \right] \int_{T_0}^{T} \int_{B_R} \left(u^{p+m-1} + u^p\right) \ |x|^{-\gamma} \, \mathrm{d}x \, \mathrm{d}t \,, \end{aligned}$$

where $c_1 \equiv C\left(m,p\right) := 2K_\psi \ c_{m,p}^{-1}$ and $K_\psi > 0$ depends on N. The proof of the inequality (1.1.2) is concluded. $\square$

Proof of the lower energy inequalities (1.1.3) **and** (1.1.4)

We will perform before a common step, used in the proof of both inequalities. Let us fix $x_0 \in \mathbb{R}^N$, and simply denote $B_R = B_R(x_0)$ when no confusion arises. We always consider $p \in \mathbb{R} \setminus \{0\}$.

• Step 1. *First energy inequality.* In this step prove the following inequality for $-p < 1 - m$, $p \neq 0$:

$$\begin{aligned} \frac{p+1}{p} & \left[\int_{B_R} \left(u(T,x)^{-p} \ \psi^2 \left(T,x\right) - u(T_0,x)^{-p} \ \psi^2 \left(T_0,x\right) \right) |x|^{-\gamma} \, \mathrm{d}x \right] \\ & + \frac{2m \left(p+1\right)^2}{\left(m-p-1\right)^2} \int_{T_0}^{T} \int_{B_R} \left| \nabla u^{\frac{m-p-1}{2}} \right|^2 \psi^2 |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t \\ & \le 2m \int_{T_0}^{T} \int_{B_R} u^{m-p-1} \ |\nabla \psi|^2 \ |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t + 2\frac{p+1}{p} \int_{T_0}^{T} \int_{B_R} u^{-p} \ \psi \ |\psi_t| \ |x|^{-\gamma} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.3.14}$$

We just sketch the proof, since it is similar - but simpler - to Step 2 of the proof of the upper energy inequality (1.1.2): we approximate $u^{-p-1}\psi^2$ with admissible test functions, we use the weak formulation of the equation (0.0.18) and after a double limiting process, we obtain:

$$\begin{aligned} -\left(p+1\right) \int_{T_0}^{T} \int_{B_R} u_t \, u^{-(p+1)} \psi^2 |x|^{-\gamma} \, \mathrm{d}x \, \mathrm{d}t + \frac{4m \left(p+1\right)^2}{\left(m-p-1\right)^2} \int_{T_0}^{T} \int_{B_R} \left| \nabla u^{\frac{m-p-1}{2}} \right|^2 \psi^2 |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t \\ \le \frac{4m \left(p+1\right)}{\left(m-p-1\right)} \int_{T_0}^{T} \int_{B_R} \psi \nabla u^{\frac{m-p-1}{2}} \cdot u^{\frac{m-p-1}{2}} \nabla \psi \ |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Using Young's inequality, i.e., $|v \cdot w| \le \frac{|v|^2}{2\varepsilon} + \frac{\varepsilon |w|^2}{2}$, with $\varepsilon = -\frac{p+1}{m-p-1} > 0$ we get the following inequality

$$\begin{aligned} \frac{p+1}{p} \int_{T_0}^{T} \int_{B_R} \partial_t (u^{-p}) \psi^2 \ |x|^{-\gamma} \, \mathrm{d}x \, \mathrm{d}t + \frac{2m \left(p+1\right)^2}{\left(m-p-1\right)^2} \int_{T_0}^{T} \int_{B_R} \left| \nabla u^{\frac{m-p-1}{2}} \right|^2 \psi^2 |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t \\ \le 2m \int_{T_0}^{T} \int_{B_R} u^{m-p-1} \ |\nabla \psi|^2 \ |x|^{-\beta} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Integrating by parts in time the first term of the above inequality, we obtain (3.3.14).

In the following steps we show that (3.3.14) implies both (1.1.3) and (1.1.4). We will just sketch the proof, since it is very similar to the proof of inequality (1.1.2).

• STEP 2. *Proof of inequality* (1.1.3). In order to keep the same notation in the proofs, we will change the sign of the exponent p with respect to the statement of inequality (1.1.3), namely we will switch p to $-p$. So $m-1<p<0$, hence we have $\frac{p+1}{p}<0$. We follow the Steps 3 of the proof of inequality (1.1.2): we choose a smooth $0\le\psi\le 1$ supported in $[T_0,T]\times B_R$, such that $\psi\equiv 1$ on $[T_0,T_1]\times B_{R_1}$, $\psi(T,x)=0$ for all $x\in B_R$ and $\psi(t,x)=0$ for all $(t,x)\in[T_0,T]\times\partial B_R$, so that there exists $K_\psi>0$ (depending only on N) such that $|\nabla\psi(t,x)|^2\le K_\psi\,(R-R_1)^{-2}$ and $|\psi_t(t,x)|\le K_\psi(T-T_1)^{-1}$ for all $(t,x)\in(T_0,T]\times B_R\setminus B_{R_1}$. With this choice of ψ we obtain the following inequality:

$$\begin{aligned}\int_{B_{R_1}(x_0)} u(T_0,x)^{-p}|x|^{-\gamma}\,\mathrm{d}x+\int_{T_0}^{T_1}\int_{B_{R_1}}|\nabla u^{\frac{m-p-1}{2}}|^2|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t\\ \le c_4\left[\int_{T_0}^{T}\int_{B_R}u^{m-p-1}|\nabla\psi|^2|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t+\int_{T_0}^{T}\int_{B_R}u^{-p}|\psi_t||x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\right].\end{aligned}$$

Proceeding as in Step 3 of the proof of inequality (1.1.2) we obtain inequality (1.1.3), with

$$c_2=K_\psi\frac{2\left(m\vee\frac{|p+1|}{|p|}\right)}{|p+1|\left(\frac{1}{|p|}\wedge\frac{2m|p+1|}{(m-p-1)^2}\right)}.$$

• STEP 3. *Proof of inequality* (1.1.4). We choose ψ as in Step 3 of the proof of inequality (1.1.2) and repeating the same estimates used there, we can estimate (3.3.14) to get

$$\begin{aligned}&\int_{B_{R_1}(x_0)} u(T,x)^{-p}\;|x|^{-\gamma}\,\mathrm{d}x+\int_{T_1}^{T}\int_{B_R}\left|\nabla u^{\frac{-p+m-1}{2}}\right|^2\;|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t\\ &\le c_3\left[\frac{h_\sigma(R,R_1,x_0)}{(R-R_1)^\sigma}+\frac{1}{T_1-T_0}\right]\int_{T_0}^{T}\int_{B_R}\left(u^{-p+m-1}+u^{-p}\right)\;|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Finally, inequality (1.1.4) follows by letting $T=\tau$ and taking the supremum in $\tau\in[T_1,T]$ in the above inequality. The constant $c_3>0$ becomes

$$c_3=\frac{4K_\psi}{p+1}\,\frac{m\vee\frac{p+1}{p}}{\frac{1}{p}\wedge\frac{4m(p+1)}{(m-p-1)^2}}>0,\qquad\text{since } p>0.$$

The proof of Lemma 1.1.1 is now concluded. □

Proof of the Caccioppoli estimates of Lemma 1.1.2.

We just sketch the proof. We use the test function ψ^2u^{-m}, assuming first $0<\delta\le u\le M$, and we approximate it as in Step 2 of the proof of the upper energy inequality (1.1.2) so that we obtain

$$\begin{aligned}-\iint_Q\psi^2\partial_t(u^{1-m})|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t+m^2\,(1-m)\iint_Q\psi^2|\nabla\log u|^2|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t\\ \le 2m\,(1-m)\iint_Q\psi\nabla\log u\cdot\nabla\psi\;|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{3.3.15}$$

where $Q = (\tau, t) \times B_R(x_0)$. Inequality (1.1.5) follows by using Young's inequality $ab \le \varepsilon a^2 + b^2/4\varepsilon$, with $\varepsilon = m/4$, on the right-hand side of (3.3.15) and integrating by parts in time the first term of inequality (3.3.15). Note that the assumption $u \in [\delta, M]$ can be removed by a lengthy but straightforward approximation, but we refrain from doing this here, since we apply (1.1.5) only to solutions to a "lifted" Dirichlet problem (δ-MDP), which we already know to be positive and bounded. $\square$

3.4 Appendix-B

The goal of this Appendix is to prove the weighted Caffarelli-Kohn-Nirenberg and Poincaré Inequalities of Propositions 0.0.10 and 0.0.11 and to provide some useful quantitative information about the auxiliary function $\rho_{x_0}^{\gamma,\beta}$ and its inverse. We recall here the expression of $\rho_{x_0}^{\gamma,\beta}$ defined in (0.0.19):

$$\rho_{x_0}^{\gamma,\beta}(R) := \left(\int_{B_R(x_0)} |x|^{(\beta-\gamma)\frac{N}{2}} \,\mathrm{d}x \right)^{\frac{2}{N}} .$$

We begin with a technical lemma on the behaviour of the function $\rho_{x_0}^{\gamma,\beta}$.

Lemma 3.4.1. *Let $N \ge 3$, assume that $\gamma, \beta \in \mathbb{R}$ satisfy* (0.0.11). *Then there exists $\overline{\kappa}_{16}, \overline{\kappa}_{18} > 0$ such that for any $y \in \mathbb{R}^N$ and for any $R > 0$ the following inequalities hold*

$$\begin{aligned} \overline{\kappa}_{16}^{-1}\, \rho_y^{\gamma,\beta}(R) \le R^2 \frac{\mu_\gamma(B_R(y))}{\mu_\beta(B_R(y))} &\le \overline{\kappa}_{16} \rho_y^{\gamma,\beta}(R)\,, \\ \overline{\kappa}_{18}^{-1} R^2 \left[R \vee |y|\right]^{\beta-\gamma} \le \rho_y^{\gamma,\beta}(R) &\le \overline{\kappa}_{18} R^2 \left[R \vee |y|\right]^{\beta-\gamma} . \end{aligned} \tag{3.4.1}$$

The constants $\overline{\kappa}_{16}, \overline{\kappa}_{18} > 0$ depend only on N, γ, β.

Proof. We will only prove the first inequality appearing in (3.4.1), the second one will follow by the same estimates, noticing that Step 2 and 3 correspond to the cases $|y| \le 2R$ and $|y| > 2R$. The proof will be divided in different cases.

- CASE 1. *Assume $y = 0$ and $R > 0$.* This case is done by a direct calculation.
- CASE 2. *Assume that* $0 < |y| \le 2R$. The reader may observe that in this case the following inclusions holds $B_r(y) \subset B_{4r}(0) \subset B_{8r}(y)$. Then, by the doubling property, we obtain the following inequalities (recall that $\sigma = 2 + \beta - \gamma$)

$$\begin{aligned} \mu_{-(\sigma-2)\frac{N}{2}}(B_R(y))^{\frac{2}{N}} \mu_\beta(B_R(y)) &\le \mu_{-(\sigma-2)\frac{N}{2}}(B_{4R}(0))^{\frac{2}{N}} \mu_\beta(B_{4R}(0)) \le C_1 R^{2+\beta-\gamma} R^{N-\beta} \\ &\le C_2 R^2 \mu_\gamma(B_{4R}(0)) \le C_3 R^2 \mu_\gamma(B_{8R}(y)) \le C_4 R^2 \mu_\gamma(B_R(y)). \end{aligned}$$

The other inequality is obtained by similar techniques.

- CASE 3. *Assume that* $0 < 2R < |y|$. Assume that $z \in B_R(y)$, therefore $\frac{|y|}{2} \le |z| \le \frac{3|y|}{2}$. In order to prove inequality (3.4.1) we will show that the quantity I defined by

$$I = \left(\frac{1}{R^N} \int_{B_R(y)} |z|^{(\sigma-2)\frac{N}{2}} \,\mathrm{d}z \right)^{\frac{2}{N}} \left(\frac{1}{R^N} \int_{B_R(y)} |z|^{-\beta} \,\mathrm{d}z \right) \left(\frac{1}{R^N} \int_{B_R(y)} |z|^{-\gamma} \,\mathrm{d}z \right)^{-1},$$

is bounded (above and below) by a constant independent of y and R. For any $\alpha > -N$ we can estimate $\int_{B_R(y)} |z|^\alpha \,\mathrm{d}z$ as

$$C_5 R^N |y|^\alpha \le \int_{B_R(y)} |z|^\alpha \,\mathrm{d}z \le C_6 R^N |y|^\alpha,$$

where the constants C_5 and C_6 depend only on the dimension N. Therefore the quantity I is bounded (above) by

$$I \le C_7 |y|^{\beta-\gamma}|y|^{-\beta}|y|^{\gamma} \le C_7,$$

recall that $\sigma = 2+\beta-\gamma$. The very same technique works also for the other bound. $\square$

The function $\rho_{x_0}^{\gamma,\beta}(r)$, is increasing in r therefore it has an inverse which we denote by $\left(\rho_{x_0}^{\gamma,\beta}\right)^{-1}$, whose behaviour we show in the next lemma.

Lemma 3.4.2. *Let $\gamma,\beta \in \mathbb{R}$ satisfy* (0.0.11) *and $N \ge 3$. Then there exists $\overline{\kappa}_{19} > 0$ such that for any $x_0 \in \mathbb{R}^N$ and for any $s > 0$ the following inequalities hold*

$$\overline{\kappa}_{19}^{-1}\, s^{\frac{1}{2}} \left[s^{\frac{1}{\sigma}} \vee |x_0|\right]^{\frac{\gamma-\beta}{2}} \le \left(\rho_{x_0}^{\gamma,\beta}\right)^{-1}(s) \le \overline{\kappa}_{19}\, s^{\frac{1}{2}} \left[s^{\frac{1}{\sigma}} \vee |x_0|\right]^{\frac{\gamma-\beta}{2}}, \tag{3.4.2}$$

where constant $\overline{\kappa}_{19} > 0$ depends only on N,γ,β. As a consequence, for any $x_0 \in \Omega \subset \mathbb{R}^N$ and for any $s \in (0,T]$, we have:

$$\left(\rho_{x_0}^{\gamma,\beta}\right)^{-1}(s) \le \begin{cases} \overline{\kappa}_{19}\, s^{\frac{1}{\sigma}}, & \text{if } \sigma \ge 2, \\ \overline{\kappa}_{19}\, s^{\frac{1}{2}} \left(T^{\frac{1}{\sigma}} \vee \sup\limits_{x_0 \in \Omega} |x_0|\right)^{\frac{\gamma-\beta}{2}}, & \text{if } 0 < \sigma < 2\,. \end{cases} \tag{3.4.3}$$

Proof. We first observe that inequality (3.4.3) easily follows by (3.4.2), hence we only have to prove the latter.
• Case 1. *Assume* $x_0 = 0$. Under this assumption we know that $\rho_0^{\gamma,\beta}(r) \asymp r^{\sigma}$. Therefore $\left(\rho_{x_0}^{\gamma,\beta}\right)^{-1}(s) \asymp s^{\frac{1}{\sigma}}$. • Case 2. *Assume* $x_0 \neq 0$. Here we deal with two different cases. First, we observe that if $0 \le r \le |x_0|$ we have $\rho_{x_0}^{\gamma,\beta}(r) \asymp r^2 |x_0|^{\beta-\gamma}$, therefore $\left(\rho_{x_0}^{\gamma,\beta}\right)^{-1}(s) \asymp s^{\frac{1}{2}}|x_0|^{\frac{\gamma-\beta}{2}}$ and the estimate holds when $r \asymp s^{\frac{1}{2}}|x_0|^{\frac{\gamma-\beta}{2}} \le |x_0|$, i.e. when $s^{\frac{1}{\sigma}} \le |x_0|$. Next, when $0 \le |x_0| \le r$ we have $\rho_{x_0}^{\gamma,\beta}(r) \asymp r^{\sigma}$ and therefore $\left(\rho_{x_0}^{\gamma,\beta}\right)^{-1}(s) \asymp s^{\frac{1}{\sigma}}$, the estimate holds when $s^{\frac{1}{\sigma}} \ge |x_0|$. The two estimates give (3.4.2), and this concludes the proof. $\square$

Proof of the weighted Poincaré inequality of Proposition 0.0.11. The Poincaré inequality (0.0.29) will easily follow from Hölder's inequality and from the following weighted Sobolev-Poincaré inequality proven in [94, Theorem I]

$$\left(\int_{B_R(y)} |\phi - \overline{\phi}|^{r^*} |x|^{-\gamma}\,\mathrm{d}x\right)^{\frac{1}{r^*}} \le C_1 R \frac{\mu_\gamma(B_R(y))^{\frac{1}{r^*}}}{\mu_\beta(B_R(y))^{\frac{1}{2}}} \left(\int_{B_R(y)} |\nabla\phi|^2 |x|^{-\beta}\,\mathrm{d}x\right)^{\frac{1}{2}}, \tag{3.4.4}$$

where $\overline{\phi} = \mu_\gamma\left(B_R(y)\right)^{-1} \int_{B_R(y)} \phi|x|^{-\gamma}\,\mathrm{d}x$, $B_R(y)$ is any ball and $C_1 > 0$ depends only on N,γ and β. $\square$

Proof of Proposition 0.0.10. Inequality (CKNI2) follows from (3.4.4), estimating the constant as in the above proof, then using $\|f - \overline{f}\|_{\mathrm{L}^p_\gamma(B_R(x_0))} \ge \|f\|_{\mathrm{L}^p_\gamma(B_R(x_0))} - \overline{f}\mu_\gamma(B_R(x_0))^{\frac{1}{p}}$ and Hölder's inequality. $\square$

The following technical lemma is needed in the proof of Proposition 3.1.2.

Lemma 3.4.3. *Let $N \ge 3$, assume that $\gamma,\beta \in \mathbb{R}$ satisfy* (0.0.11). *For any positive real number A,*

$$A \ge 4 \vee 2\overline{\kappa}_{18} \vee \left(4\overline{\kappa}_{18}^2\right)^{\frac{1}{\sigma}}, \tag{3.4.5}$$

and for any $r>0$, for any $x_0\in\mathbb{R}^N$ the following inclusion holds

$$Q_{R/A}(t_0,x_0)\subset Q_R^+(t_0,x_0), \tag{3.4.6}$$

where Q_R^+ and Q_R are defined in (3.1.3), *and $\overline{\kappa}_{18}>0$ is as in* (3.4.1).

Proof. We prove only in the case $0<\sigma<2$, namely $\gamma>\beta$, since the case $\sigma\ge 2$ is actually simpler and follows by the very same steps. In order to prove the inclusion (3.4.6) we need to verify two conditions: $2R/A\le R/2$ and $4\rho_{x_0}^{\gamma,\beta}(R/A)\le\rho_{x_0}^{\gamma,\beta}(R)$. The first condition is automatically verified by (3.4.5), hence we only need to verify the latter, which easily follows by the following estimates:

$$4\rho_{x_0}^{\gamma,\beta}(R/A)\le 4\overline{\kappa}_{18}\frac{R^2}{A^2}\left[\frac{R}{A}\vee|x_0|\right]^{-(\gamma-\beta)}\le\frac{4\overline{\kappa}_{18}^2}{A^\sigma}\overline{\kappa}_{18}^{-1}R^2\left[\frac{R\vee|x_0|}{A}\right]^{-(\gamma-\beta)}\le\rho_{x_0}^{\gamma,\beta}(R)\,,$$

which follow from $\frac{R\vee|x_0|}{A}\le\frac{R}{A}\vee|x_0|$ together with the condition (3.4.5). The proof is concluded. $\square$

Further estimates on test functions

The operator $\mathcal{L}_{\gamma,\beta}f=|x|^\gamma\nabla\cdot\left(|x|^{-\beta}\nabla f\right)$ acts on smooth functions as follows:

$$\mathcal{L}_{\gamma,\beta}(\phi)=|x|^{\gamma-\beta}\left[\Delta\phi-\beta\frac{x}{|x|^2}\cdot\nabla\phi\right]. \tag{3.4.7}$$

In the proof of Proposition 1.2.1, we use the following technical Lemma.

Lemma 3.4.4. *For any $x_0\in\mathbb{R}^N$ and any $R>0$ there exists $\phi\in C_c^2(\mathbb{R}^N)$ such that* $\mathrm{supp}(\phi)\subset B_{2R}(x_0)$, *$\phi\equiv 1$ on $B_R(x_0)$, $0\le\phi\le 1$ and the following estimate holds:*

$$\phi^{\frac{-m}{1-m}}(x)|\mathcal{L}_{\gamma,\beta}(\phi)(x)|^{\frac{1}{1-m}}\le\overline{\kappa}_{10}\left(\rho_{x_0}^{\gamma,\beta}(R)\right)^{-\frac{1}{1-m}}\qquad\text{for all } x\in\mathbb{R}^N\,, \tag{3.4.8}$$

where $\overline{\kappa}_{10}>0$ depends only on N,γ,β and m.

Proof. We define a function $\phi:=\psi\left(|x-x_0|^\sigma R^{-\sigma}\right)^b$ with $b>0$ to be chosen later; we choose the cutoff function $\psi:[0,\infty)\to[0,1]$ to be smooth. A simple calculation shows that

$$\begin{aligned}\nabla\phi&=b\sigma R^{-\sigma}\psi\left(|x-x_0|^\sigma R^{-\sigma}\right)^{b-1}\psi'\left(|x-x_0|^\sigma R^{-\sigma}\right)|x-x_0|^{\sigma-2}(x-x_0),\\ \Delta\phi&=b\sigma R^{-\sigma}\psi^{b-2}|x-x_0|^{\sigma-2}\left[\sigma|x-x_0|^\sigma R^{-\sigma}\left((b-1)|\psi'|^2+\psi\psi''\right)+\psi\psi'(N+\beta-\gamma)\right].\end{aligned}$$

Using the expression (3.4.7) we get

$$\begin{aligned}\phi^{\frac{-m}{1-m}}\left|\mathcal{L}_{\gamma,\beta}(\phi)\right|^{\frac{1}{1-m}}&=\psi^{\frac{-bm}{1-m}}\left|\mathcal{L}_{\gamma,\beta}\left(\psi^b\right)\right|^{\frac{1}{1-m}}=\psi^{\frac{-bm+(b-2)}{1-m}}\left[b\sigma R^{-\sigma}|x|^{2-\sigma}|x-x_0|^{\sigma-2}\right]^{\frac{1}{1-m}}\\ &\quad\times\left|\sigma|x-x_0|^\sigma R^{-\sigma}\left((b-1)|\psi'|^2+\psi\psi''\right)+\psi\psi'(N+\beta-\gamma)-\psi\psi'\beta|x|^{-2}(x-x_0)\cdot x\right|^{\frac{1}{1-m}}.\end{aligned}$$

We need to split the proof in two cases, depending on the relation between $|x_0|$ and R.
• *When* $0\le|x_0|\le\frac{3}{2}R$*:* Choosing $\psi=\psi(|x-x_0|)$ to be a equal to 1 on $B_{(7/4)R}(x_0)$ and zero outside $B_{2R}(x_0)$, we have $\mathrm{supp}(\mathcal{L}_{\gamma,\beta}(\psi^b))\subseteq B_{(7/4)R}(x_0)^c\cap B_{2R}(x_0)$; since $B_{(1/4)R}(0)\subset B_{(7/4)R}(x_0)$, it turns out that $\mathrm{supp}(\mathcal{L}_{\gamma,\beta}(\psi^b))\subseteq\{(1/4)R\le|x|\le 3R\}\cap\{(7/4)R\le|x-x_0|\le 2R\}$. Taking $b\ge\frac{2}{1-m}$ we obtain

$$\phi^{\frac{-m}{1-m}}\left|\mathcal{L}_{\gamma,\beta}(\phi)\right|^{\frac{1}{1-m}}\le C_\sigma\left[b\sigma R^{-\sigma}\right]^{\frac{1}{1-m}}\left[\sigma 4^\sigma(|b-1||\psi'|^2+|\psi''|)+|\psi'|(N+\beta-\gamma+4|\beta|)\right]^{\frac{1}{1-m}},$$

where we have used that $0 \le \psi \le 1$; note that $C_\sigma > 0$ depends only on σ and, by (3.4.1) in this case $\rho_{x_0}^{\gamma,\beta}(R) \asymp R^\sigma$, this proves (3.4.8).

• *When* $|x_0| \ge \frac{3}{2}R$*:* In this case we choose $\psi(|x - x_0|)$ equal to 1 on $B_R(x_0)$ and equal to 0 outside $B_{(5/4)R}(x_0)$. In this way, $\text{supp}(\mathcal{L}_{\gamma,\beta}(\psi^b)) \subseteq \{R \le |x - x_0| \le (5/4)R\} \subseteq \{(1/3)|x_0| \le |x| \le (11/6)|x_0|\}$; noticing that $\{R \le |x - x_0| \le (5/4)R\} \subset \{|x| \ge R/4\}$, using (3.4.1) and proceeding as in the previous case, we conclude the proof of (3.4.8). $\square$

Part II

Global Harnack Principle and Asymptotic Behaviour of Fast Diffusion Equation and Fast p-Laplace Equation

Introduction to Part II

In Part II we use the quantitative estimates obtained in Part I to study the large-time behaviour and the behaviour for large $|x|$ of nonnegative solutions to nonlinear singular diffusion equations with weights. In Chapter 4 we consider non-negative solutions to the Cauchy-Problem for the Weighted Fast Diffusion Equation (WFDE)

$$\begin{cases} \partial_t u = |x|^\gamma \nabla \cdot \left(|x|^{-\beta} \nabla u^m \right) & \text{in } (0, \infty) \times \mathbb{R}^d, \\ u\,(0, x) = u_0(x) & \text{in } \mathbb{R}^d\,, \end{cases} \tag{CP}$$

where

$$u_0 \in \mathrm{L}^1_{\gamma,+}(\mathbb{R}^d) = \{u_0 : \mathbb{R}^d \to \mathbb{R} : u_0 \geq 0\,, \int_{\mathbb{R}^d} u_0\,|x|^{-\gamma}\,\mathrm{d}x < \infty\}\,,$$

and $m \in (m_c, 1)$, where $m_c := \frac{d-2-\beta}{d-\gamma}$. Along Part II d shall always be the dimension of $\mathbb{R}^d$ and we shall always consider $m \in (m_c, 1)$. We will always consider the following range of parameters,

$$\gamma < d \qquad \text{and} \qquad \gamma - 2 < \beta \leq \frac{d-2}{d}\gamma\,.$$

This range is optimal for the validity of a family of the so-called Caffarelli-Kohn-Nirenberg inequalities, see [5] and Part I. The nonlinear operator $|x|^\gamma \nabla \cdot \left(|x|^{-\beta} \nabla u^m \right)$ was introduced in the 80s by Kamin and Rosenau [6, 7, 8] to model heat propagation -or more generally singular/degenerate diffusion- in inhomogeneous media; the parabolic problem has been studied by many authors since then, mostly in the case $m \geq 1$ and with only one weight [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47].

For (CP), existence, uniqueness and a comparison principle have been proven in [18, 86] and in the range $m_c < m < 1$ conservation of mass holds. In the non-weighted case $\gamma = \beta = 0$, the WFDE becomes the standard Fast Diffusion Equation (FDE) which has been intensively studied in recent years by many authors: it is hopeless to give here a complete bibliography, hence we refer to the monographs [48, 4] and [49, 25] for a complete account, as well as for the physical relevance of the model.

In Chapter 5 we investigate the initial value problem

$$\begin{cases} u_t(t, x) = \Delta_p u(t, x) & \text{in } (0, \infty) \times \mathbb{R}^d, \\ u\,(0, x) = u_0(x) & \text{in } \mathbb{R}^d\,, \end{cases} \tag{p-CP}$$

in the fast diffusion range $\frac{2N}{N+2} < p < 2$. The initial data is supposed to be $0 \leq u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and the p-Laplace operator is defined as

$$\Delta_p w := \nabla \cdot \left(|\nabla w|^{p-2} \nabla w \right)\,.$$

Equation $u_t = \Delta_p u$ has been extensively studied and has several applications, for instance in non-Newtonian mechanics, see [10], the limit case $p \to 1$ has particular importance in the treatment of images, see [11]. Existence and uniqueness of solution to (p-CP) has been settled in [105], the comparison principle holds. For a complete account we refer to the monograph [78], [48, Chapter 11] and references therein.

The aim of Part II is to provide a global picture of the fine behaviour of the solutions to both (CP) *and* (p-CP)*, classified in a precise way in terms of the initial data.*

Though to fix ideas and avoid technical complications we shall discuss here the problem (CP) in the un-weighted case (i.e. $\gamma = \beta = 0$), what we will say holds for the general case $\gamma \neq 0, \beta \neq 0$ and for solutions to (p-CP) as well. As we already mentioned in the case $\gamma = \beta = 0$, the equation $u_t = |x|^\gamma \nabla \cdot \left(|x|^{-\beta} \nabla u^m\right)$ becomes the standard Fast Diffusion Equation,

$$u_t = \Delta u^m \,, \qquad \text{where} \quad \frac{d-2}{d} < m < 1 \,, \tag{FDE}$$

if m is in the range considered in (FDE) then mass conservation holds for all data in $\mathrm{L}^1(\mathbb{R}^d)$. In what follows we will denote $\mathrm{L}^1_{0,+}(\mathbb{R}^d)$ as $\mathrm{L}^1_{+}(\mathbb{R}^d)$. It is well known (see [4]) that (FDE) admits a family of self-similar solutions called Barenblatt solutions, given by

$$\mathfrak{B}_M(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0 \frac{t^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_1 |x|^2\right]^{\frac{1}{1-m}}} \tag{3.4.9}$$

where $\vartheta^{-1} = 2 - d(1-m) > 0$, b_0, b_1 are constants which depends only on d, m and M is the mass (i.e. $M = \|\mathfrak{B}_M\|_{\mathrm{L}^1(\mathbb{R}^d)}$) of $\mathfrak{B}_M$. We shall often write $\mathfrak{B}_M(t)$ to denote the function $t \to \mathfrak{B}_M(t,\cdot)$. The solution $\mathfrak{B}_M$ is also called "fundamental solution" since it satisfies the following relation

$$\lim_{t \to 0} \mathfrak{B}_M(t,x) = M\,\delta_0 \,,$$

where the above limit is understood in the sense of distributions. It is also known, see for instance [106, 107, 108, 109], that solutions $u(t,x)$ to (CP) with initial data $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ converge to the self-similar profile $\mathfrak{B}_M$, with the same mass as u_0, in $\mathrm{L}^1(\mathbb{R}^d)$-norm, namely

$$\|u(t) - \mathfrak{B}_M(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \to 0 \qquad \text{where} \quad M = \int_{\mathbb{R}^d} u_0 \,\mathrm{d}x \,. \tag{3.4.10}$$

The above statement is sometimes called the *Central Limit Theorem* for integrable solution to (FDE) (see [110, 109]), making an analogy between convergence of non-negative, integrable solutions to the (FDE) towards the Barenblatt profile and the convergence to the *Gaussian* law

$$G(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \tag{3.4.11}$$

in the central limit theorem for the Heat Equation (HE)

$$u_t = \Delta u \,, \tag{HE}$$

see [111] and references therein.

Here we consider solutions to (FDE) with nonnegative, integrable data and we want to further investigate the phenomenon of convergence towards $\mathfrak{B}_M$ and see how far we can go beyond the known convergence result (3.4.10). We want to stress that this is already a non-trivial question in the case of (HE), indeed a wide zoology of solutions is available. The Gaussian G attracts only integrable data, the non zero constant solutions are probably the easiest example of solutions which do not converge to G, solutions with initial data $u_0 \in \mathrm{L}^p(\mathbb{R}^d) \setminus \mathrm{L}^1(\mathbb{R}^d)$ also have a different behaviour, see [111]. If we restrict the study on $\mathrm{L}^1(\mathbb{R}^d)$ data, it is a natural question to ask whether the convergence towards the Gaussian holds in a stronger norm, for instance the L^∞ or maybe in the sense of *uniform convergence* of the *relative error*, namely

$$\left\| \frac{u(t,x) - G(t,x)}{G(t,x)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \to 0 \qquad \text{as} \qquad t \to \infty\,. \tag{UCRE}$$

In general, such a strong convergence does not hold for solutions to (HE). Indeed, it was proven in [112] that solutions to (HE) with nonnegative, integrable initial data that satisfy

$$V_0(x) \sim \frac{A}{|x|^\alpha} \qquad \text{as} \quad |x| \to \infty\,, \text{ for some} \quad \alpha > d\,,$$

produce solutions that behave with the same power-like decay, namely

$$V(t,x) \sim C(t)\,|x|^{-\alpha} \qquad \text{as} \quad |x| \to \infty\,.$$

The relative error $V(t,x)/G(t,x)$ is unbounded and uniform convergence of the relative error does not take place. The problem seems to be in the tails (i.e. the behaviour of the initial data for large $|x|$), indeed, there is no uniform control of the tails of the above solution $V(t,x)$ in terms of $G(t,x)$ in the whole $\mathbb{R}^d$, i.e.

there are no finite $C_1, C_2 > 0$ *such that*

$$C_1\, G(t,x) \le V(t,x) \le C_2\, G(t,x) \qquad \textit{for any} \quad x \in \mathbb{R}^d\,. \tag{3.4.12}$$

Nevertheless, inequality (3.4.12) holds on bounded domains and it goes under the name of *Harnack Principle* or *Harnack Inequality*, see Part I for more information about Harnack inequalities in the nonlinear setting. More of such examples and other curious phenomena related to the (HE) can be found in [111].

Global Harnack Principle (GHP) We call GHP an estimate (from below and from above) of any nonnegative integrable solution $u(t,x)$ in terms of suitable Barenblatt profiles (see for instance inequality (3.4.14)). For solutions to (FDE), the *Global Harnack Principle* is known under strong assumptions. The following result was proven in [110].

Theorem (GHP, [110])**.** *Let $u(t,x)$ be a solution to* (CP) *with initial data $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. Assume that there exists $R > 0$ and $A > 0$ such that*

$$u_0(x) \le \frac{A}{|x|^{\frac{2}{1-m}}}\,, \qquad \textit{for any} \quad |x| \ge R\,. \tag{3.4.13}$$

Then for any $t_0 > 0$ there exists $\tau_1, \tau_2 > 0$ and $M_1, M_2 > 0$ such that

$$\mathfrak{B}_{M_1}(t - \tau_1, x) \le u(t,x) \le \mathfrak{B}_{M_2}(t + \tau_2, x) \qquad \textit{for any} \quad t \ge t_0\,,\ x \in \mathbb{R}^d\,. \tag{3.4.14}$$

Estimates similar to (3.4.14) were also proven in the context of bounded domains, see [113], and of Riemannian manifolds, see [54, 56]. One can look at the GHP as a suitable generalization of the Harnack inequalities proven in Part I for local solutions in the context of (CP).

We will see in Chapter 4 that (3.4.13) is not only sufficient, but also very close to being necessary. Indeed, if we do not assume (3.4.13) we can construct the following counterexample. Let $m > \frac{d}{d+2}$, and consider the solution $w(t,x)$ to (CP) with initial data w_0 given by

$$w_0(x) = \frac{1}{(1+|x|^2)^{\frac{m}{1-m}}}\,.$$

It is clear that w_0 does not satisfy assumption (3.4.13) and, for $|x|$ large enough, we have that $w_0(x) > \mathfrak{B}_M(t_0,x)$ for any $t_0, M > 0$. However, $w_0 \in \mathrm{L}^1(\mathbb{R}^d)$ for any $m > \frac{d}{d+2}$. As will be clear below the behaviour of $w(t,x)$ does not resemble the one given by the Barenblatt profile. Indeed, by constructing explicit sub/super-solutions, we are able to show that for any time $t>0$

$$\frac{1}{\left((4m(1-m)d\,t+1)^{\frac{1}{1-m}}+|x|^2\right)^{\frac{m}{1-m}}} \le w(t,x) \le \frac{(1+t)^{\frac{m}{1-m}}}{(1+t+|x|^2)^{\frac{m}{1-m}}}\,. \tag{3.4.15}$$

The inequality above gives us remarkable insight about the long time behaviour of the solution $w(t,x)$. First, for any time $t>0$, $w(t,x)$ has a power-like behaviour at infinity, namely that $w(t,x) \sim |x|^{\frac{-2m}{1-m}}$ as $|x| \to \infty$, which differs substantially from the Barenblatt one. Second, in view of this anomalous tail behaviour, an inequality as (3.4.14) is simply not possible, due to the not-matching powers of the tail of $w(t,x)$ with respect to the Barenblatt one.

Coming back to the (FDE), we are able now to state the main question that we want to address in Part II.

What is the largest class of nonnegative, integrable solutions which satisfy the GHP (inequality (3.4.14)*)?* (Q1)

We want to stress here that inequality (3.4.14) gives us a very big amount of information: such estimates allow a uniform control of the tail and prove infinite speed of propagation, they allow to prove a uniform quantitative control of $C^\alpha(\mathbb{R}^d)$ (for $\alpha \in (0,1)$) and $C^k(\mathbb{R}^d)$ norms, they imply uniform convergence of the relative error and ultimately give us a quantitative insight on how the solution approaches the Barenblatt profile. Therefore, we believe that it is of paramount importance to give a precise answer to the above question.

Our main result will be a complete answer to the above question in both (CP) and (p-CP) cases. Indeed we will say more, we will prove a sort of *Generalized Global Harnack Principle*, which classifies the tail behaviour of solutions to (CP) in terms of the power-like decay of the initial data.

Let us first comment on the main obstructions in proving inequality (3.4.14). In [99] it was already proven that nonnegative, integrable solutions to (FDE) develop at least Barenblatt-like tails. We shall prove the same kind of result for solutions to (CP) and to to (p-CP) (see Theorem (4.2.1) in Chapter 4, Theorem (5.2.2) in Chapter 5). Therefore the only obstruction is represented by the upper bound of inequality (3.4.14). The sharp condition on the initial data u_0 in order to obtain such an upper bound turns out to be a decay condition, but not of the kind assumed in (3.4.13), indeed we need an integral kind of decay: to be more precise let us define the space $\mathcal{X}$.

Let $f \in \mathrm{L}^1(\mathbb{R}^d)$, $m \in (\frac{d-2}{d},1)$, we say that $f \in \mathcal{X}$ or equivalently that it satisfies the *tail-condition* (TC) if

$$|f|_{\mathcal{X}} := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{B_R^c(0)} |f(x)|\,\mathrm{d}x < \infty\,, \tag{TC}$$

with a slight abuse of language we will also call $\mathcal{X}$ the tail space. Recall that for $m \in (\frac{d-2}{d}, 1)$ we have that $\frac{2}{1-m} - d > 0$. It is easily seen that $|\cdot|_{\mathcal{X}}$ is a norm. Intuitively the quantity $|f|_{\mathcal{X}}$ measures *how fast* the function f decays at ∞ relatively to the decay of the Barenblatt profile $\mathfrak{B}_M$. We now introduce a subspace of $\mathrm{L}^1(\mathbb{R}^d)$ of functions that satisfy the tail condition (TC), that will play a key role in the rest of the thesis:

$$\mathcal{X} := \{u \in \mathrm{L}^1(\mathbb{R}^d) : |u|_{\mathcal{X}} < +\infty\}. \tag{3.4.16}$$

In the general case ($\gamma \neq \beta \neq 0$) the space $\mathcal{X}$ depends on γ, β and it is defined in Chapter 4, see Chapter 5 for the (p-CP) case. We observe that $\mathcal{X}$ contains functions which do not satisfy the assumption (3.4.13), we postpone the construction of such examples to section (4.5) of Chapter 4, nevertheless we have the following

Theorem 3.4.5. *Let $u(t, x)$ be a solution to* (CP) *with an initial data $0 \leq u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. Then the GHP (inequality* (3.4.14)*) holds if and only if*

$$u_0 \in \mathcal{X} \setminus \{0\}\,.$$

The above Theorem provides necessary *and* sufficient conditions for the GHP to hold, indeed, it gives a complete answer to (Q1).

Generalized Global Harnack Principle In section (4.4.3) in Chapter 4 we will prove an inequality similar to (3.4.14) but for initial data which do not belong to $\mathcal{X}$. Indeed, by means of constructing super/sub-solutions we shall prove that if the initial data u_0 has a power-like decay for $|x| \to \infty$ such as

$$(A + |x|)^{-\alpha} \leq u_0(x) \leq (B + |x|)^{-\alpha} \qquad \text{where} \quad d < \alpha < 2/(1-m)$$

then it will produce a solution $u(t, x)$ with the same power-like behaviour for $|x| \to \infty$ for all times, so no change in the qualitative behaviour at infinity occurs. This is a remarkable fact since it shows a clear difference between $\mathcal{X}$ and $\mathcal{X}^c$, indeed it suggests that the space $\mathcal{X}$ is stable for the fast diffusion flow. We can say more, our results provide a precise characterization of the fine behaviour of solutions in terms of the initial data and allows to see a clear separation between two classes of data, in terms of the space $\mathcal{X}$. Such a separation is better understood if we pass to *self-similar* variables.

Self similar variables. Nonlinear Fokker-Plank equation. Let $u(t, x)$ be a solution to (CP) with initial data u_0, and consider $R(t) = \left(\frac{t+1}{\vartheta}\right)^{\vartheta}$, where $\vartheta^{-1} = 2 - d(1-m)$. The *self-similar change of variables*

$$v(\tau, y) = \frac{R(t)^d}{\zeta^d}\, u(t, x) \quad \text{where} \quad \tau = \frac{1}{\sigma} \log \frac{R(t)}{R(0)}\,,\ y = \frac{\zeta\, x}{R(t)}\,, \zeta = \left(\frac{1-m}{2m}\right)^{\vartheta} \tag{3.4.17}$$

transforms $u(t, x)$ into a solution to the following nonlinear *Fokker-Planck* type equation

$$\frac{\partial v}{\partial \tau} + \mathrm{div}\left(v\, \nabla v^{m-1}\right) = 2\, \mathrm{div}\,(\, v\, x)\ , \tag{NFPE}$$

with initial data $v_0(y) = \frac{\zeta^{d-\gamma}}{R(0)^{d-\gamma}} u_0(\frac{\zeta\, x}{R(0)})$. Notice that the mass of the initial data is unchanged. Finally we recall that the Barenblatt profile $\mathfrak{B}_M(t, x)$ is transformed into the stationary profile

$$\mathcal{B}_M(y) = \frac{1}{(C(M) + |y|^2)^{\frac{1}{1-m}}}\,.$$

where $C(M)$ is a constant which depends on the mass M and on d and m. In *self-similar* variables the nonlinear version of the central limit theorem (3.4.10) can be restated as

$$\|v(\tau) - \mathcal{B}_M\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \tau \to \infty\,,$$

where again M is the mass of the initial data v_0. Lastly let us define the Manifold of the Barenblatt solutions:

$$\mathcal{M}_{\mathcal{B}} := \{\mathcal{B}_M \ : \ M > 0\}\ , \tag{3.4.18}$$

and the distance

$$\mathrm{d}_1(f) := \inf_{\mathcal{B}_M \in \mathcal{M}_{\mathcal{B}}} \|f - \mathcal{B}_M\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}\,.$$

The (NFPE) **as a flow in** $\mathrm{L}^1_+(\mathbb{R}^d)$ **and the space** $\mathcal{X}$**.** It is well-known that the (NFPE) can be seen as the gradient flow of an Entropy functional, see [114, 115, 116, 117]. Solutions to (NFPE) can be seen as continuous paths that will eventually converge to a point of the manifold $\mathcal{M}_{\mathcal{B}}$, this is another interpretation of the Central Limit Theorem (3.4.10). Hence, the *basin of attraction* in the d_1 topology (or L^1 topology), of the manifold $\mathcal{M}_{\mathcal{B}}$ is the whole space $\mathrm{L}^1_+(\mathbb{R}^d)$. However here appears an interesting issue: we can split $\mathrm{L}^1_+(\mathbb{R}^d)$ in two disjoint sets

$$\mathrm{L}^1_+(\mathbb{R}^d) = \mathcal{X} \cup \mathcal{X}^c\,,$$

with a slight abuse of language, we identify $\mathcal{X}$ with the space $\mathcal{X} \cap \mathrm{L}^1_+(\mathbb{R}^d)$. As we will see in Chapter 4 the flow is stable in both sets, i.e., if $u_0 \in \mathcal{X}$ then the solution stays in $\mathcal{X}$, namely $u(t) \in \mathcal{X}$ for all $t > 0$, and if $u_0 \in \mathcal{X}^c$ then the solution $u(t) \in \mathcal{X}^c$ and there is no crossing from $\mathcal{X}$ to $\mathcal{X}^c$ and viceversa, see figure 3.1 for an illustration of this fact.

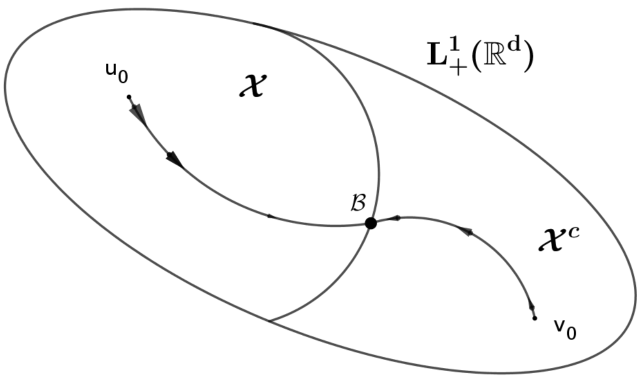

Figure 3.1: We represent two possible paths in $\mathrm{L}^1_+(\mathbb{R}^d)$, one starting in $\mathcal{X}$ and one in $\mathcal{X}^c$, there are no crossing lines between $\mathcal{X}$ and $\mathcal{X}^c$. We notice that the manifold of the Barenblatt solutions $\mathcal{M}_{\mathcal{B}}$ is contained in the topological boundary (with respect to the L^1 topology) of $\mathcal{X}$: $\mathcal{M}_{\mathcal{B}} \subset \partial_{\mathrm{L}^1}\mathcal{X}$.

Surprisingly the key feature that allows to discriminate different behaviours is a fine analysis of the tails, that is better seen through uniform convergence in relative error topology, that is

$$\mathrm{d}_\infty(f) := \inf_{\mathcal{B}_M \in \mathcal{M}_{\mathcal{B}}} \left\|\frac{f}{\mathcal{B}_M} - 1\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\,.$$

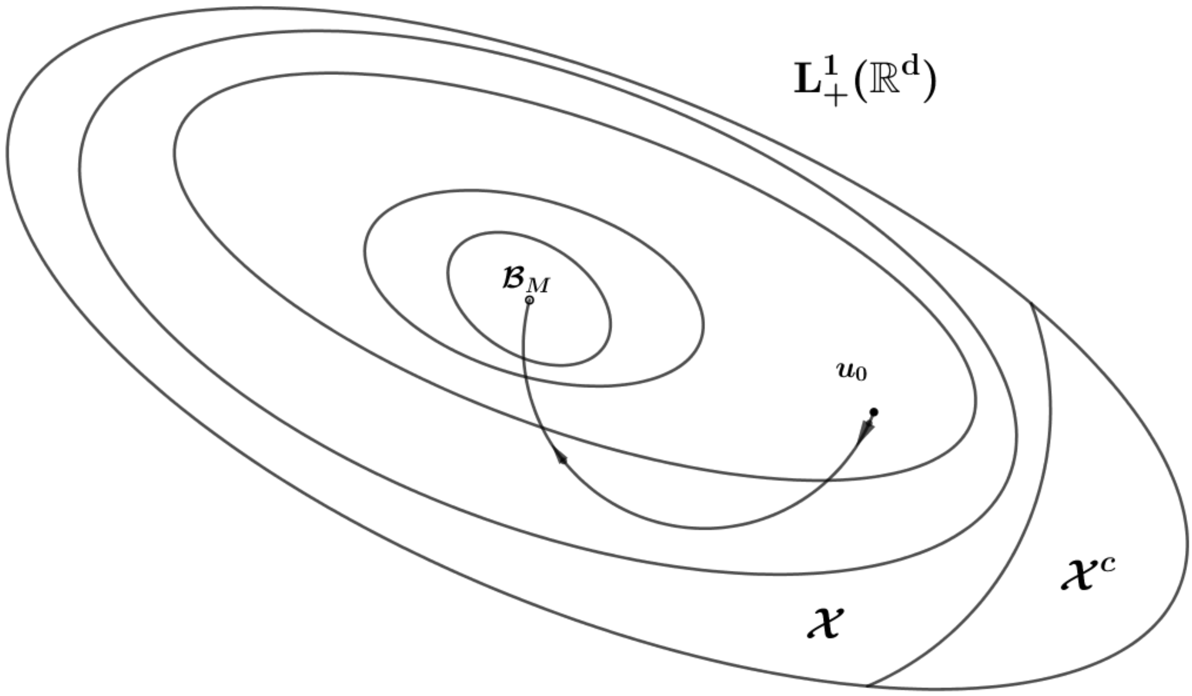

Figure 3.2: Illustration of the stability of the sets $\mathcal{X}_r$. Once the solution enters one of those sets it will stay there.

The first thing to notice is that, even if $v(\tau, y) \to \mathcal{B}_M(y)$ as $\tau \to \infty$ a.e., hence also in relative error, i.e. $v(\tau, y)\mathcal{B}_M^{-1}(y) \to 1$ as $\tau \to \infty$ a.e., the uniform convergence of the relative error may fail. Indeed, it is not hard to see that what happens for the solution of $w(t, x)$ in (3.4.15) is that

$$\left\| \frac{w(\tau)}{\mathcal{B}_M} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = \infty \qquad \text{for all } \tau > 0\,,$$

and uniform convergence of the relative error does not take place. It is then quite natural to ask

What is the largest class of nonnegative, integrable solutions for which uniform convergence of the relative error takes place? (Q2)

One of the main results of Chapter 4 solves this issue: again $\mathcal{X}$ is the correct space we are looking for. Let us state the main theorem that solves the issue, we will state it in original variables.

Theorem 3.4.6. *Let* $m \in \left(\frac{d-2}{d}, 1\right)$ *and let* u *be a solution to* (CP) *with initial data* $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *and* $M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}$. *Then* $u(t, x)$ *converges to the Barenblatt profile* $\mathfrak{B}_M(t, x)$ *in* uniform relative error*, i.e.*

$$\lim_{t\to\infty} \left\| \frac{u(t, x) - \mathfrak{B}_M(t, x)}{\mathfrak{B}_M(t, x)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0 \tag{3.4.19}$$

if and only if the initial data u_0 *satisfies the tail condition* (TC)*, namely*

$$u_0 \in \mathcal{X}^+ \setminus \{0\}$$

The above theorem provides an explicit characterization that allows to discriminate wether a solution will converge or not to $\mathcal{B}$ (or equivalently to the manifold $\mathcal{M}_{\mathcal{B}}$) uniformly in relative error. We have stated it for the particular case $\gamma = \beta = 0$ but it holds also in the general setting. This explains the two disjoint subsets on $\mathrm{L}^1_+(\mathbb{R}^d)$, namely $\mathcal{X}$ and $\mathcal{X}^c$. Let us analyze the possible different scenarios for the flow in $\mathrm{L}^1_+(\mathbb{R}^d)$. Before starting our analysis, we just mention that from now on we will use the terms “uniform convergence of the relative error” and “convergence in uniform relative error” as equivalent and both will refer to the validity of (3.4.19).

- *If* $v_0 \in \mathcal{X}^c$. In this case $d_\infty(v_0) = \infty$ holds unconditionally. Notice that $\mathcal{M}_{\mathcal{B}} \subset \mathcal{X}$. We know that the solution $v(\tau) \in \mathcal{X}^c$ since $v_0 \in \mathcal{X}^c$. Indeed we can show that

$$v_0 \in \mathcal{X}^c \Longrightarrow \left\| \frac{v(\tau)}{\mathcal{B}_M} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = +\infty \qquad \text{for all } \tau > 0 \Longrightarrow v(\tau) \in \mathcal{X}^c \,.$$

  This tells us that if an initial datum v_0 has a tail strictly above the Barenblatt one, then the corresponding solution will have a tail at least of the same size as v_0, for all times. This is the case, for instance, for the solution $w(t,x)$ considered in (3.4.15). In particular it reveals that it is impossible to have bounds of the form

$$\mathrm{d}_\infty(v(\tau)) \le \mathrm{d}_1(v(\tau))\,,$$

  if we do not have it already (at least) for the initial datum. Hence the two distances d_1 and d_∞ are really different along the flow, since any $u_0 \in \mathrm{L}^1_+$ is sent by the flow to a unique element of $\mathcal{M}_{\mathcal{B}}$ in the L^1 topology, but not in uniform relative error. Indeed in the latter case, the flow stays always at infinite d_∞-distance from $\mathcal{M}_{\mathcal{B}}$.

- *If* $v_0 \in \mathcal{X}$ *and* $\mathrm{d}_\infty(v_0) < \infty$. Recall that $\mathcal{M}_{\mathcal{B}} \subset \mathcal{X}$, and that it is the set of stationary solutions or equilibria of (NFPE). The GHP tells us that

$$v_0 \in \mathcal{X} \Longrightarrow \left\| \frac{v(\tau)}{\mathcal{B}_M} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \infty \qquad \text{for all } \tau > 0 \Longrightarrow v(\tau) \in \mathcal{X} \Longrightarrow d_\infty(v(\tau)) \xrightarrow[\tau\to\infty]{} 0\,.$$

  Indeed the GHP implies a stability result for the flow, since it can be rewritten as

$$\mathrm{d}_\infty(v(\tau)) \le F(|v_0|_{\mathcal{X}}, \|v_0\|_{\mathrm{L}^1(\mathbb{R}^d)})\,,$$

  for some locally bounded function F. In other words, if we begin close to the manifold B, we stay close to it, see figure 3.2 for an illustration of this fact. In this case it is possible to prove uniform convergence of the relative error, a stronger statement than merely a convergence in d_1-distance (or L^1 topology). Indeed, a finer analysis can be done, we discuss briefly this issue below.

- *If* $v_0 \in \mathcal{X}$ *and* $\mathrm{d}_\infty(v_0) = \infty$*.* This is the borderline case. The GHP implies a strong control of the tail, which means

$$\mathrm{d}_\infty(v(\tau)) < \infty \qquad \text{for any } \tau > 0\,,$$

  and so we can essentially apply the analysis of the previous scenario. This case was not known in the literature and shows how sharp our result is.

Finer analysis in $\mathcal{X}$. Let us define $\mathcal{X}^\sharp$ and for any $r > 0$ the set $\mathcal{X}^\sharp_r$ as

$$\mathcal{X}^\sharp = \{v \in \mathcal{X} : \mathrm{d}_\infty(v) < \infty\} \qquad \text{and} \qquad \mathcal{X}^\sharp_r = \{v \in \mathcal{X}^\sharp : \mathrm{d}_\infty(v) < r\}\,,$$

it follows that

$$\mathcal{X} = \mathcal{X}^{\sharp} \cup \left(\mathcal{X}^{\sharp}\right)^c \qquad \text{and} \qquad \mathcal{X}^{\sharp} = \bigcup_{r>0} \mathcal{X}^{\sharp}_r \,.$$

The GHP implies stability of the flow in $\mathcal{X}$ and $\mathcal{X}^{\sharp}$: for any $v_0 \in \mathcal{X}$ there exists $r_0, \tau_0 > 0$ s.t. $d_\infty(v(\tau)) < r_0$ for all $\tau \geq \tau_0$, hence the flow never exits a certain $\mathcal{X}^{\sharp}_{r_0}$. Indeed, we show more: $d_\infty(v(\tau)) \to 0$ as $\tau \to \infty$, which means that the flow always leaves the manifolds $d_\infty(v(\tau)) = r$ (level sets of distance from $\mathcal{M}_\mathcal{B}$) to enter one at a lower level, say $d_\infty(v(\tau)) = r - \varepsilon$, see fig. (3.2) for an illustration of this fact.

This can be summarized as follows: we show that the solution map S_τ sends immediately $\mathcal{X}$ into the more regular $\mathcal{X}^{\sharp}$

$$S_\tau : \mathcal{X} \to \mathcal{X}^{\sharp} \,.$$

Also, notice that $\lim_{\tau\to\infty} S_\tau(X) = \mathcal{M}_\mathcal{B}$ holds in the sense of uniform relative error if and only if $u_0 \in \mathcal{X}$, while $\lim_{\tau\to\infty} S_\tau(\mathrm{L}^1_+(\mathbb{R}^d)) = \mathcal{M}_\mathcal{B}$ holds in the $\mathrm{L}^1(\mathbb{R}^d)$ topology.

Rates of Convergence. The natural question that we address here is: are there "universal rates" of convergence towards $\mathcal{M}_\mathcal{B}$? More precisely:

in self-similar variables, can we find a speed of convergence to the stationary profile which is valid for all solutions starting from data in $\mathcal{X}$?

In different words, can we find a function $f(\tau)$ for which $f(\tau) \to 0$ as $\tau \to \infty$ such that

$$\left\| \frac{v(\tau)}{\mathcal{B}_M} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq f(\tau)$$

In [118], Carrillo and Vázquez computed the *rate of convergence* for radial initial data which satisfy the decay property (3.4.13). They have proven that

$$\left\| \frac{u(t,x) - \mathfrak{B}_M(t,x)}{\mathfrak{B}_M(t,x)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = O(t^{-1}) \,, \tag{3.4.20}$$

for all the range $\frac{d-2}{d} < m < 1$. Notice that the above result is stated in original variables. In particular, again in [118], Carrillo and Vázquez leave unanswered the following question

What is the largest class of initial data for which (3.4.20) *holds?* (Q3)

The answer to the above questions is delicate and cannot be easily given for all $m \in (0,1)$, neither for all $m \in (\frac{d-2}{d}, 1)$. Some preliminary remarks are in order. In the case $\gamma = \beta = 0$ the question has a long history (see [119]): for $\frac{d-2}{d} < m < 1$ under suitable assumptions, it has been proven in [108, 116, 120, 12, 121, 117, 122], with techniques that involve the relative entropy functional (introduced in [123, 124]) or exploiting the so-called Bakry-Émery methods, (see [125]), that there exist (sharp) rates of convergence in different topologies, the most common being the d_1. The rate t^{-1} computed above was extended to a larger class of data in [126]. In a quite long series of papers, [127, 128, 129, 53] similar results were obtained in the whole range $0 < m < 1$, we recall that in the range $0 < m < \frac{d-2}{d}$ there is a dramatic change in the behaviour of solutions since they may vanish in finite time, see [48, 4]. In the general case $\gamma \neq 0, \beta \neq 0$, rates of convergence were obtained in [18, 19].

In Chapter 4 we will show how we can combine the techniques of this paper with the ones used in [128, 53, 129], to obtain rates of convergence to the Barenblatt profile with an (almost) uniform rate in the whole $\mathcal{X}$. For reasons that are not entirely clear up to now, we need to restrict ourselves to the range $\frac{d-1}{d} = m_1 < m < 1$ in the case $\gamma = \beta = 0$, and to the range $\frac{2d-2-\beta-\gamma}{2(d-\gamma)} < m < 1$, $\gamma < 0$ for the general case, see [18, 19] for further remarks. The former restriction is somehow natural, since, at least when $\gamma = \beta = 0$, the FDE is a gradient flow of a displacement convex functional (the relative entropy) with respect to the so-called Wasserstein distance, see [114, 117, 115]. The displacement convexity is lost below m_1. The main result reads:

Theorem (Almost sharp universal rates of convergence in the non-weighted case). *Let u be the solution to* (CP) *corresponding to the initial data $0 \le u_0 \in \mathcal{X}$, $\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = M$, $\int_{\mathbb{R}^d} x u_0(x) \, \mathrm{d}x = 0$ and assume that $\beta = \gamma = 0$. Then, for every $\delta \in (0, 1)$ there exist $t_\delta, c_\delta > 0$ (that may also depend on u_0) such that for all $t > t_\delta$*

$$\|u(t) - \mathfrak{B}_M(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}} \qquad \textit{and} \qquad t^{d\vartheta} \|u(t) - \mathfrak{B}_M(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}}.$$

Remark. Notice that the above result is new for the whole space $\mathcal{X}$ even if we are dealing with the case $\gamma = \beta = 0$. Indeed, all the previous results deal with more restrictive assumption such as radial data, a very precise control for $|x| \to \infty$ or being sandwiched between two Barenblatt profiles.

When dealing with CKN-weights, the result is a bit weaker, because of the possible lack of C^k regularity at the origin and reads:

Theorem (Almost sharp universal rates of convergence in the weighted case). *Let u be the solution to* (CP) *corresponding to the initial data $0 \le u_0 \in \mathcal{X}$ with $\int_{\mathbb{R}^d} u_0 |x|^{-\gamma} \, \mathrm{d}x = M$ and assume $\gamma < 0$. Then, there exists a $\delta_* \in (0, 1)$ such that for every $\delta \in (0, \delta_*)$ there exist $t_\delta, c_\delta > 0$ (that may also depend on u_0) such that for all $t > t_\delta$*

$$\|u(t) - \mathfrak{B}_M(t)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}} \qquad \textit{and} \qquad t^{(d-\gamma)\vartheta} \|u(t) - \mathfrak{B}_M(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}}.$$

In the above Theorem $\vartheta^{-1} = (2 + \beta - \gamma) - (d - \gamma)(1 - m)$. If we consider radial initial data in $\mathcal{X}$ we can provide a *universal* rate of convergence, very much in the spirit of [118] or [126].

Theorem (Sharp universal rates for radial data). *Assume $\gamma = \beta = 0$ and let $m \in \left(\frac{d-2}{d}, 1\right)$. Let u be the solution to* (CP) *corresponding to the radial initial data $0 \le u_0 \in \mathcal{X}$, with $\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = M$. Then, there exist $t_0, c_0 > 0$ (that may also depend on u_0) such that for all $t > t_0$*

$$\left\| \frac{u(t)}{\mathfrak{B}_M(t)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_0}{t}. \tag{3.4.21}$$

Remark. As an immediate consequence of (3.4.21) we obtain that for all $t \ge t_0$

$$\|u(t) - \mathfrak{B}_M(t)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \le \frac{c_0}{t} \qquad \text{and} \qquad t^{(d-\gamma)\vartheta} \|u(t) - \mathfrak{B}_M(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_0}{t}.$$

The above Theorem solves a problem left open in [118], i.e. identifying the largest class of non-negative *radial* L^1 data for which the above rate of convergence holds and gives an answer to (Q3). Such rates are proven to be sharp, since they are fulfilled by two time-shifted Barenblatt, with the same mass, see [118, 126]. Finally, we observe that, even if we restrict the analysis to radial data, the class $\mathcal{X}$ is much larger than those considered up to now in the literature: we refer to Section 4.5

of Chapter 4 for examples of functions in $\mathcal{X}$ which do not resemble the behaviour of the Barenblatt profile.

On the p-Laplace Case. Almost all of what we have said holds true for solutions to (p-CP). However, we shall remark that there are some differences. The p-Laplace case happens to be less studied in the regime $\frac{2d}{d+1} < p < 2$ so fewer results are available. In Chapter 5 we have proven the counterpart of Theorem 3.4.5 and of Theorem 3.4.6, however, up to know, rates of convergence in uniform relative error are not available. We have not constructed solutions that have a tail behaviour substantially different from the Barenblatt profile but we remark that it can be done. Finally, we want to observe that in Section 5.3.3 we have used an interesting connection (discovered in [130]) between radial solutions to (CP) and radial solutions to (p-CP), this characterization allows us to give a quantitative description in space (for large $|x|$) of the gradient of solutions to (p-CP).

Stability of Gagliardo-Nirenberg Inequalities. In Chapter 6 our purpose is to establish a new stability result for a special class of subcritical Gagliardo-Nirenberg inequalities. We develop a new strategy for studying the stability, in which the flow of the fast diffusion equation is used as a tool. Let us consider the Gagliardo-Nirenberg-Sobolev inequalities (GNS in what follows)

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\rm GN} \, \|f\|_{2p} \quad \forall\, f \in \mathcal{D}(\mathbb{R}^d) \tag{3.4.22}$$

where $\mathcal{D}(\mathbb{R}^d)$ denotes the set of smooth functions on $\mathbb{R}^d$ with compact support. The exponent p is in the range $(1, +\infty)$ if $d = 1$ or 2, and $p \in \big(1, d/(d-2)\big]$ if $d \geq 3$. The exponent $\theta = \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$ is determined by the scaling invariance. According to [12],

$$\mathsf{g}(x) = \big(1 + |x|^2\big)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

is an optimal function of (3.4.22), and the set of all optimal functions is the manifold of the functions $g_{\lambda,\mu,y}(x) := \mu\, \mathsf{g}\left((x-y)/\lambda\right)$ parametrized by $(\lambda, \mu, y) \in (0, +\infty) \times (0, +\infty) \times \mathbb{R}^d$. When $p = d/(d-2)$, this manifold is the set of the Aubin-Talenti functions.

Inequality (3.4.22) can also be written in non-scale invariant form as

$$\delta[f] := (p-1)^2 \, \|\nabla f\|_2^2 + 4\,\frac{d - p\,(d-2)}{p+1}\, \|f\|_{p+1}^{p+1} - 2\,\mathcal{K}_{\rm GN}\, \|f\|_{2p}^{2p\,\frac{d\,(p-1) - 2\,(p+1)}{d\,(p-1) - 4\,p}} \geq 0 \tag{3.4.23}$$

and equality is again achieved by g. See [131, Section 4.1] for details on how $\mathcal{C}_{\rm GN}$ and $\mathcal{K}_{\rm GN}$ are related. We will call δ the *deficit functional*.

In this chapter we study the *stability* properties of the GNS. The main question we want to address here is the following:

$$\textit{Assume that } \delta[f] \textit{ is small, in what sense, if any, is } f \textit{ close to } \mathsf{g}? \tag{Q4}$$

The issue of *stability* of optimal functions in the Calculus of Variations started with the study of solitary waves obtained by minimization methods as in [132, 133, 134]. In recent years, the problem of finding stability estimates for the sharp inequalities both in analysis and geometry such as the isoperimetric inequality, the Brunn–Minkowski inequality, the Sobolev inequality, the logarithmic Sobolev inequality, etc, was intensively studied.

In the case of Sobolev inequality and GNS some early results have been obtained in bounded domains in [135, 136], but the result of G. Bianchi and H. Egnell for the critical Sobolev inequality [13] was immediately recognized as a major breakthrough, with the irritating drawback that the constant is still unknown. In the critical case $p=d/(d-2)=2^*/2$, $d\geq 3$, G. Bianchi and H. Egnell proved in [13] the existence of a positive constant $\mathcal{C}$ such that

$$\frac{4}{(d-2)^2}\,\|\nabla f\|_2^2-2\,\mathcal{K}_{\rm GN}\,\|f\|_{2^*}^2\geq\mathcal{C}\,\inf\|\nabla f-\nabla g\|_2^2\,,$$

where the infimum is taken over the $(d+2)$-dimensional manifold of the Aubin-Talenti functions. However, the existence of $\mathcal{C}$ is obtained by contradiction and no quantitative estimate of $\mathcal{C}$ has been obtained so far.

Our goal is establish a *quantitative* analogue of the estimate of G. Bianchi and H. Egnell in the subcritical range $p\in(1,2^*/2)$. More specifically, we aim at proving that $\delta[f]$ controls a distance to the function g under some suitable assumptions.

Before stating our main result, we need to introduce the *relative entropy functional*, let $d\geq 3$ and $p\in\big(1,d/(d-2)\big]$,

$$\mathcal{E}[f]=\left(\frac{2p}{1-p}\right)\int_{\mathbb{R}^d}\left(|f|^{p+1}-\mathsf{g}^{p+1}-\frac{1+p}{2p}\mathsf{g}^{1-p}(|f|^{2p}-\mathsf{g}^{2p})\right)dx\,,$$

it may not appear obvious but $\mathcal{E}$ is a positive functional. Indeed, $\mathcal{E}$ is naturally associated to the Fast Diffusion Flow and in the next section the relation between δ and $\mathcal{E}$ will be clarified. It is interesting to notice that if $\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}=\|\mathsf{g}\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}$ then, by the Csiszár-Kullback inequality, the entropy controls the L^{2p} distance between f and g, namely there exists a constant $C_p>0$ such that

$$\|f-\mathsf{g}\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p}\leq C_p\,\mathcal{E}[f]^{\frac12}\,,$$

for further information see [17, 137, 138].

Let us denote $\mathrm{W}^{1,2}(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ that have a square-integrable distributional gradient. We are finally in the position of stating our main result.

Theorem 3.4.7. *Let $d\geq 3$ and and $p\in\big(1,d/(d-2)\big)$. Let $f\in\mathrm{W}^{1,2}(\mathbb{R}^d)$ such that for some $A,B>0$ we have*

$$\sup_{r>0}r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r}f^{2p}\,dx\leq A<\infty\qquad\text{and}\qquad\mathcal{E}[f]\leq B<\infty\,.$$

Suppose also that

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}=\|\mathsf{g}\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\qquad\text{and}\qquad\int_{\mathbb{R}^d}x\,f^{2p}\,dx=0\,.$$

Then there exists a constant $\mathcal{C}>0$ depending only on d,p and A,B such that

$$\delta[f]\geq\mathcal{C}\,\mathcal{E}[f]\tag{3.4.24}$$

Remark. At a first glance it may appear unnatural to consider the entropy of the function f. However, it will be clear in section 6.1.1, f^{2p} will be nothing else than the density (or solution) of the Fast Diffusion Flow, so it makes sense to consider f^2p to compute the remainder term in (3.4.23).

The reader will also recognize the tail condition in the definition of the space $\mathcal{X}$ considered in the previous chapters and now restated as

$$\sup_{r>0}r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r}f^{2p}\,dx\leq A\,.\tag{3.4.25}$$

Once again, in what follows, f^{2p} will be the density of the Fast Diffusion Flow, since we will use the Global Harnack Principle proven in Chapter 4 it is natural to ask f^{2p} to belong to the space $\mathcal{X}$. Let us clearly state that (3.4.25) cannot be dropped and it is key in our method.

Lastly, let us remark that condition (3.4.25) guarantees that both the quantities $\int_{\mathbb{R}^d} |x| f^{2p}\,\mathrm{d}x$, $\int_{\mathbb{R}^d} |x|^2 f^{2p}\,\mathrm{d}x$ are finite.

As a consequence of Theorem 3.4.7 we obtain the following Bianchi-Egnell type inequality.

Corollary 3.4.8. *Under the assumptions of Theorem 3.4.7 assume further that*

$$\|f^{2p}\,|x|^2\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|\mathsf{g}^{2p}\,|x|^2\|_{\mathrm{L}^1(\mathbb{R}^d)} \qquad \textit{and} \qquad \delta[f] \le 1\,.$$

Then there exists a constant $\mathcal{C}_2 > 0$ which depends only on d, p and on A of Theorem 6.1.4 such that

$$\delta[f] \ge \mathcal{C}_2\,\frac{\|\nabla f - \nabla g\|^8_{\mathrm{L}^2(\mathbb{R}^d)}}{\left(1 + \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}\right)^4}\,. \tag{3.4.26}$$

Both Theorem 3.4.7 and Corollary 3.4.8 answer to question Q4, however the results are quite different. Inequality (3.4.24) of Theorem 3.4.7 affirms that the deficit controls the L^{2p} distance from f to g, while inequality (3.4.26) proves stability at the level of gradients which, as often mentioned in the literature, is the strongest possible norm in this context.

Chapter 4

Global Harnack Principle for FDE with Caffarelli-Kohn-Nirenberg Weights

In this chapter we investigate the large-time behaviour and the quantitative behaviour for large $|x|$ of nonnegative solutions to nonlinear singular diffusion equations with weights. We prove sharp, global lower bounds for solution to the Cauchy Problem with non-negative, integrable initial data. If the initial data also satisfies an additional condition (called the *tail condition*), we prove that such lower estimates are matched by global upper estimates and the so called *Global Harnack Principle* holds. As a consequence, we prove that the convergence to the Barenblatt profile in uniform relative error holds if and only if the initial data satisfy the aforementioned tail condition. Finally we further investigate this property proving that it is stable under the flow given by the WFDE.

4.1 Introduction and main results

In this chapter we study the large-time behaviour and the quantitative behaviour for large $|x|$ of nonnegative solutions to nonlinear singular diffusion equations with weights as those considered in Part I. We consider non-negative solutions to the Cauchy-Problem for the Weighted Fast Diffusion Equation

$$\begin{cases} \partial_t u = |x|^\gamma \nabla \cdot \left(|x|^{-\beta} \nabla u^m\right) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u\,(0,x) = u_0(x) & \text{in } \mathbb{R}^d\,, \end{cases} \tag{CP}$$

where

$$u_0 \in \mathrm{L}^1_{\gamma,+}(\mathbb{R}^d) = \{u_0 : \mathbb{R}^d \to \mathbb{R} : u_0 \ge 0\,, \int_{\mathbb{R}^d} u_0\,|x|^{-\gamma}\,\mathrm{d}x < \infty\}\,,$$

and $m \in (m_c, 1)$, where $m_c := \frac{d-2-\beta}{d-\gamma}$. We will always consider the following range of parameters,

$$\gamma < d \qquad \text{and} \qquad \gamma - 2 < \beta \le \frac{d-2}{d}\gamma\,.$$

This range is optimal for the validity of a family of the so-called Caffarelli-Kohn-Nirenberg inequalities, see [5] and Part I. The nonlinear operator $|x|^\gamma \nabla \cdot \left(|x|^{-\beta}\nabla u^m\right)$ was introduced in the 80s by Kamin and Rosenau [6, 7, 8], to model heat propagation -or more generally singular/degenerate diffusion- in inhomogeneous media; the parabolic problem has been studied by many authors since then, mostly in the case $m \ge 1$ and with only one weight, see the Introduction to Part II.

For (CP) existence, uniqueness and a comparison principle have been proven in [18, 86] and in the range $m_c < m < 1$ conservation of mass holds. In the non-weighted case $\gamma = \beta = 0$, the WFDE becomes the standard Fast Diffusion Equation (FDE) which has been intensively studied in the recent years by many authors: it is hopeless to give here a complete bibliography, hence we refer to the monographs [48, 4] and [49, 25] for a complete account, as well as for the physical relevance of the model.

Barenblatt Solutions. The Problem (CP) admits a family of fundamental solutions of *self-similar type*, the so called Barenblatt solutions:

$$\mathfrak{B}_{M,T}(t,x) = \frac{\zeta^{d-\gamma}}{R_\star(t+T)^{d-\gamma}}\,\mathfrak{B}_M\left(\frac{\zeta\, x}{R_\star(t+T)}\right) = \frac{(t+T)^{\frac{1}{1-m}}}{\left[b_0\frac{(t+T)^{\sigma\vartheta}}{M^{\sigma\vartheta(1-m)}} + b_1|x|^\sigma\right]^{\frac{1}{1-m}}}\,, \tag{B}$$

where M is the *mass* of the solution and T is a free parameter. The profile $\mathcal{B}_M$ is given by

$$\mathcal{B}_M(y) = (C(M) + |y|^\sigma)^{\frac{1}{m-1}} \tag{4.1.1}$$

and

$$\sigma := 2+\beta-\gamma\,, \qquad \frac{1}{\vartheta} = (d-\gamma)(m-m_c)\,, \qquad \zeta^{\frac{1}{\vartheta}} = \frac{1-m}{\sigma\, m}\,. \tag{4.1.2}$$

The time rescaling $R_\star(t)$ is defined as follows

$$R_\star(t) = \left(\frac{t}{\vartheta}\right)^\vartheta\,. \tag{4.1.3}$$

The constants b_0, b_1 depend only on d, m, γ, β while $C(M)$ depend on the mass of the profile. In what follows we shall frequently use the solution (B) with the parameter $T = 0$. We will denote such a solution in the following way

$$\mathfrak{B}(t,x;M) = \mathfrak{B}_{M,0}(t,x)\,, \qquad \text{and recall that} \qquad \mathfrak{B}(0,x;M) = M\delta_0\,. \tag{4.1.4}$$

As well we shall often write $\mathfrak{B}(t;M)$ to denote the function $t \to \mathfrak{B}(t,\cdot;M)$. We introduce next a useful parameter which plays the role of an artificial dimension, to emphasize formal analogies among exponents, in the weighted and non-weighted case (i.e. when $\mathfrak{n} = d$)

$$\mathfrak{n} = \frac{2(d-\gamma)}{2+\beta-\gamma}\,, \qquad \text{so that} \qquad m_c = \frac{\mathfrak{n}-2}{\mathfrak{n}}\,. \tag{4.1.5}$$

The space $\mathcal{X}$. Let $f \in \mathrm{L}^1_\gamma(\mathbb{R}^d)$, $m \in (m_c, 1)$, we say that $f \in \mathcal{X}$ or equivalently that it satisfies the *tail-condition* (TC) if

$$|f|_{\mathcal{X}} := \sup_{R>0} R^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)} \int_{B^c_R(0)} |f(x)||x|^{-\gamma}\,\mathrm{d}x < \infty\,. \tag{TC}$$

With a slight abuse of language we will also call $\mathcal{X}$ the tail space. Recall that for $m \in (m_c, 1)$ we have that $\frac{2+\beta-\gamma}{1-m} - (d-\gamma) > 0$. It is easily seen that $|\cdot|_{\mathcal{X}}$ is a norm. Intuitively the quantity $|f|_{\mathcal{X}}$ measures *how fast* the function f decays at ∞ relatively to the decay of the Barenblatt profile $\mathfrak{B}_M$. We now introduce a subspace of $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ of functions that satisfy the tail condition (TC), that will play a key role in the rest of the paper:

$$\mathcal{X} := \{u \in \mathrm{L}^1_\gamma(\mathbb{R}^d) : |u|_{\mathcal{X}} < +\infty\}. \tag{4.1.6}$$

Main Results. The aim of this chapter is to show a global picture of the fine behaviour of the solutions to (CP), classified in a precise way in terms of the initial data. Our first task will be to understand if the Barenblatt solution is representative in any way of the *intermediate asymptotic* and of the behaviour for large $|x|$ of solutions to (CP). Since the prominent work of Herrero and Pierre, see [99], it was known that solutions to (CP) with non-negative initial data become instantaneously positive and develop a tail which is at least the one of the Barenblatt profile: a solution $u(t,x) \gtrsim |x|^{-\frac{\sigma}{1-m}}$ for large $|x|$. It was also already known (at least in the case $\gamma = \beta = 0$) that solutions do not develop a bigger tail if we consider initial data which have the same decay at infinity as the one given by the Barenblatt profile, namely $u_0(x) \lesssim |x|^{-\frac{\sigma}{1-m}}$, see for instance [110]. Our first main result is to prove that the decay condition $u_0(x) \lesssim |x|^{-\frac{\sigma}{1-m}}$ is not sharp: indeed we can consider a bigger class of data which develop the same tail of the Barenblatt profile, such a class is exactly the space $\mathcal{X}$.

Theorem 4.1.1 (Global Harnack Principle)**.** *Let u be a solution to* (CP) *with $0 \le u_0 \in \mathcal{X} \setminus \{0\}$ and let $t_0 > 0$. Then there exists $\overline{\tau}, \underline{\tau} > 0$ and $\overline{M}, \underline{M} > 0$ such that we have*

$$\mathfrak{B}(t-\underline{\tau}, x; \underline{M}) \le u(t,x) \le \mathfrak{B}(t+\overline{\tau}, x; \overline{M}), \quad \textit{for any} \quad x \in \mathbb{R}^d \textit{ and } t \ge t_0 . \tag{4.1.7}$$

The upper bound of inequality (4.1.7) *holds if and only if $u_0 \in \mathcal{X}$.*

Remark 4.1.2. The proof of the above result will be split into two cases: the upper bound, Theorem 4.3.1, and the lower bound, Theorem 4.2.1. For the upper bound the hypothesis $0 \le u_0 \in \mathcal{X}$, i.e. that u_0 has a precise decay at infinity, is strictly necessary. Indeed, for data $u_0 \notin \mathcal{X}^+$ we are able to construct explicit (sub)solutions that provide precise lower bounds that clearly contradict the upper bound of formula (4.1.7). More precisely, for any $t > 0$ and for any $x \in \mathbb{R}^d$ we have that

$$u(t,x) \ge \frac{1}{(D(t) + |x|^\sigma)^{\frac{1}{1-m} - \varepsilon}} \gneqq \mathfrak{B}(t,x;M),$$

where $\varepsilon > 0$ is small, and $D(t) \sim t^{\frac{2}{\varepsilon(1-m)}}$.
On the other hand, such hypothesis is not necessary for the lower bound of formula (4.1.7): indeed, lower bounds hold for any data $0 \le u_0 \in \mathrm{L}^1_{\gamma,\mathrm{loc}}(\mathbb{R}^d)$, see Theorem 4.2.1.

It turns out then that the Barenblatt solution $\mathfrak{B}_M$ is representative of the global behaviour (for large $|x|$) of solution to (CP) whose initial data belong to $\mathcal{X}$. We can look at this fact also in terms of convergence to the Barenblatt profile. Such a convergence has been studied by many researches and many results are available (at least in the case $\gamma = \beta = 0$). The weakest form of convergence is in L^1_γ topology, see [106], which holds for any data $0 \le u_0 \in \mathrm{L}^1_\gamma(\mathbb{R}^d)$, indeed

$$\lim_{t\to\infty} \|u(t,\cdot) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} = 0 .$$

In [106] the authors proved uniform convergence on expanding sets on the form $|x| \le Ct^\vartheta$, for the case $\gamma = \beta = 0$, namely

$$\lim_{t\to 0} \sup_{x\in\{|x|\le C\, t^\vartheta\}} \left| \frac{u(t,x) - \mathfrak{B}(t,x;M)}{\mathfrak{B}(t,x;M)} \right| = 0 , \tag{4.1.8}$$

under the condition of positive initial data $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^2(\mathbb{R}^d)$, we will prove a very similar result for the weighted case see, Theorem 4.2.4. Lately Vázquez in [107] has completed the proof of the previous result for the whole class of positive initial data which belong to $\mathrm{L}^1(\mathbb{R}^d)$. He also shows uniform convergence in $\mathrm{L}^\infty(\mathbb{R}^d)$

$$\lim_{t\to\infty} t^{d\vartheta}\|u(t,x)-\mathfrak{B}(t,x;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}=0\,. \tag{4.1.9}$$

It is a natural question to ask whether a stronger convergence in the form of *uniform relative error* may hold, i.e. can we prove under suitable assumptions that

$$\lim_{t\to\infty}\left\|\frac{u(t,x)}{\mathfrak{B}(t,x;M)}-1\right\|_{\mathrm{L}^\infty}=0, \tag{4.1.10}$$

where $\mathfrak{B}(t,x;M)$ is the Barenblatt solution with the same mass of u_0? Before starting our analysis, let us just mention that from now on we will use the terms "uniform convergence of the relative error" and "convergence in uniform relative error" as equivalent and both will refer to the validity of (4.1.10). The answer of the above question is not easy, since, as we already mentioned in Remark 4.1.2 there exist solutions which do not share the Barenblatt behaviour for large $|x|$. For such solutions indeed it happens that

$$\left\|\frac{u(t,x)}{\mathfrak{B}(t,x;M)}-1\right\|_{\mathrm{L}^\infty}=\infty \qquad \text{for any} \quad t>0\,,$$

so convergence of uniform relative error does not take place. However, in [107] Vázquez has proven that under the hypothesis $0\le u_0\lesssim |x|^{-\frac{2}{1-m}}$ (that paper considers only the case $\gamma=\beta=0$) convergence in uniform relative error does hold. Indeed the threshold seems to be again the space $\mathcal{X}$. In section 4.3 we will prove the following result.

Theorem 4.1.3. *Let u be a solution to* (CP) *with initial data $0\le u_0\in\mathrm{L}^1_\gamma(\mathbb{R}^d)\setminus\{0\}$ and $M=\|u_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}$. Then $u(t,x)$ converges to the Barenblatt profile $\mathfrak{B}(t,x;M)$ in* uniform relative error, *i.e.*

$$\lim_{t\to\infty}\left\|\frac{u(t,x)-\mathfrak{B}(t,x;M)}{\mathfrak{B}(t,x;M)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}=0 \tag{4.1.11}$$

if and only if the initial data u_0 satisfies the tail condition (TC)*, namely*

$$u_0\in\mathcal{X}\setminus\{0\}\,.$$

Once the result of convergence in uniform relative error is established we can talk about *rates of convergence.* In [118], Carrillo and Vázquez considered the case $\gamma=\beta=0$ and computed the rate of convergence for radial initial data which satisfy the decay property $u_0\lesssim|x|^{-\frac{2}{1-m}}$, they have proven that

$$\left\|\frac{u(t,x)-\mathfrak{B}(t,x;M)}{\mathfrak{B}(t,x;M)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}=O(t^{-1})\,, \tag{4.1.12}$$

for all the range $\frac{d-2}{d}<m<1$. In particular, again in [118], Carrillo and Vázquez leave unanswered the following question

What is the largest class of solutions for which (4.1.12) *holds?*

Many have contributed to this problem, see for instance [126, 128], more will be discussed in subsection 4.3.5. Here we prove that the *largest class* of initial data for which the *uniform* convergence in relative error takes place is $\mathcal{X}$, providing a partial answer to the open problem posed by Carrillo and Vázquez.

The main result reads:

Theorem 4.1.4 (Almost sharp universal rates of convergence in the non-weighted case)**.** *Let u be the solution to* (CP) *corresponding to the initial data $0 \le u_0 \in \mathcal{X}\backslash\{0\}$, $\int_{\mathbb{R}^d} u_0 \,\mathrm{d}x = M$, $\int_{\mathbb{R}^d} x u_0(x)\,\mathrm{d}x = 0$ and assume that $\beta = \gamma = 0$. Then, for every $\delta \in (0,1)$ there exist $t_\delta, c_\delta > 0$ (that may also depend on u_0) such that for all $t > t_\delta$*

$$\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}} \qquad \textit{and} \qquad t^{d\vartheta}\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}}.$$

Remark 4.1.5. Notice that the above result is new for the whole space $\mathcal{X}$ even if we are dealing with the case $\gamma = \beta = 0$. Indeed, all the previous results deal with more restrictive assumption as radial data, a very precise control for $|x| \to \infty$ or being sandwiched between two Barenblatt profiles.

When dealing with CKN-weights, the result is a bit weaker, because of the possible lack of C^k regularity at the origin and reads:

Theorem 4.1.6 (Almost sharp universal rates of convergence in the weighted case)**.** *Let u be the solution to* (CP) *corresponding to the initial data $0 \le u_0 \in \mathcal{X} \setminus \{0\}$ with $\int_{\mathbb{R}^d} u_0|x|^{-\gamma}\,\mathrm{d}x = M$ and assume $\gamma < 0$. Then, there exists a $\delta_* \in (0,1)$ such that for every $\delta \in (0,\delta_*)$ there exist $t_\delta, c_\delta > 0$ (that may also depend on u_0) such that for all $t > t_\delta$*

$$\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}} \qquad \textit{and} \qquad t^{(d-\gamma)\vartheta}\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}}.$$

If we consider radial initial data in $\mathcal{X}$ we can provide a *universal* rate of convergence, very much in the spirit of [118] or [126].

Theorem 4.1.7 (Sharp universal rates for radial data)**.** *Assume $\gamma = \beta = 0$ and let $m \in \left(\frac{d-2}{d}, 1\right)$. Let u be the solution to* (CP) *corresponding to the radial initial data $0 \le u_0 \in \mathcal{X} \setminus \{0\}$, with $\int_{\mathbb{R}^d} u_0\,\mathrm{d}x = M$. Then, there exist $t_0, c_0 > 0$ (that may also depend on u_0) such that for all $t > t_0$*

$$\left\|\frac{u(t)}{\mathfrak{B}(t;M)} - 1\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_0}{t}. \tag{4.1.13}$$

Remark 4.1.8. As an immediate consequence of (4.1.13) we obtain that for all $t \ge t_0$

$$\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \le \frac{c_0}{t} \qquad \text{and} \qquad t^{(d-\gamma)\vartheta}\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_0}{t}.$$

The above Theorem solves a problem left open in [118], i.e. identifying the largest class of nonnegative *radial* L^1 data for which the above rate of convergence holds. Such rates are proven to be sharp, since they are fulfilled by two time-shifted Barenblatt, with the same mass, see [118, 126]. Finally, we observe that, even if we restrict the analysis to radial data, the class $\mathcal{X}$ is much larger than those considered up to know in the literature: we refer to Section 4.5 for examples of functions in $\mathcal{X}$ which do not resemble the behaviour of the Barenblatt profile.

The flow in $\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$ and the space $\mathcal{X}$. It is well-known that solutions to (CP) can be seen as continuous paths, however here appears an interesting issue: we can split $\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$ in two disjoint sets

$$\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d) = \mathcal{X} \cup \mathcal{X}^c\,.$$

With a slight abuse of language, we identify $\mathcal{X}$ with the space $\mathcal{X} \cap \mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$. As we will see in section 4.5 the flow is stable in both sets, i.e., if $u_0 \in \mathcal{X}$ then the solution stays in $\mathcal{X}$, namely $u(t) \in \mathcal{X}$

for all $t > 0$, and if $u_0 \in \mathcal{X}^c$ then the solution $u(t) \in \mathcal{X}^c$ and there is no crossing from $\mathcal{X}$ to $\mathcal{X}^c$ and viceversa, see also Introduction to Part II.

Organization of this chapter. We will split the proofs of our main results in two sections. In Section 4.2 we consider the general case of $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ data and we shall prove the infinite speed of propagation, the lower bound of inequality (4.1.7) and the convergence in relative error in parabolic domains (set of the form $|x| \le Ct^\vartheta$). In section 4.3 we shall focus on initial data which belong to $\mathcal{X}$, proving the upper bound of inequality (4.1.7), the *uniform* convergence of the relative error, giving the proof of the aforementioned rates. In Section 4.4 we shall construct the afore mentioned counterexamples to the uniform convergence in relative error and we shall prove, for some particular data, a form of *Generalized* global Harnack principle giving a quantitative control of the tails. Finally, in section 4.5 we shall consider the flow in $\mathcal{X}$ and give detailed information about the space $\mathcal{X}$.

Definition of solution. We already discussed in Part I what are the main obstructions in defining a weighted Sobolev space. Here we just define $H^1_{\gamma,\beta}\left(\mathbb{R}^d\right)$ to be the closure of $C^\infty\left(\mathbb{R}^d\right)$ with the topology given by the norm $\|\phi\|^2_{H^1_{\gamma,\beta}(\mathbb{R}^d)} = \|\phi\|^2_{2,\gamma} + \|\nabla\phi\|^2_{2,\beta}$, and $\mathcal{D}_{\gamma,\beta}\left(\mathbb{R}^d\right)$ to be the closure of $C^\infty_c\left(\mathbb{R}^d\right)$ under the norm $\|\phi\|_{\mathcal{D}_{\gamma,\beta}(\mathbb{R}^d)} = \|\nabla\phi\|_{2,\beta}$, see section (0.0.1) of Part I. As a consequence, solutions to (CP) need to be considered in a suitable sense, as follows.

Definition 4.1.9. *A solution to* (CP) *is a measurable function* $u : [0,\infty) \to \mathbb{R}$ *such that*

$$u \in C_{\mathrm{loc}}([0,T);\mathrm{L}^2_{\gamma,\mathrm{loc}}(\mathbb{R}^d)) \qquad \textit{and} \qquad u^m \in \mathrm{L}^2_{\mathrm{loc}}([0,T);H^1_{\gamma,\beta,\mathrm{loc}}(\mathbb{R}^d))\,, \qquad \textit{for any} \quad 0<T<\infty$$

and the following identity holds,

$$\begin{aligned}\int_\Omega &[u(t_2,x)\phi(t_2,x) - u(t_1,x)\phi(t_1,x)]\,|x|^{-\gamma}\,\mathrm{d}x\\ &= \int_{t_1}^{t_2}\int_\Omega u\phi_t\,|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_\Omega \nabla u^m\cdot\nabla\phi\,|x|^{-\beta}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.1.14}$$

for any open $\Omega \subset \mathbb{R}^d$ *and for any interval* $[t_1,t_2] \subset [0,T]$ *and for any test function* ϕ *such that*

$$\phi \in W^{1,2}_{\mathrm{loc}}([0,T];\mathrm{L}^2_\gamma(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}([T_0,T];\mathcal{D}_{\gamma,\beta}(\Omega))$$

and we consider u_t *in the following sense*

$$u_t \in \mathrm{L}^1_{\mathrm{loc}}([0,T]\,;\,\mathrm{L}^1_{\gamma,\mathrm{loc}}(\Omega)).$$

Self-similar variables. Nonlinear Fokker-Plank equation. Since we will extensively use a particular change of variables, we have decided to define this transformation here. Let $u(t,x)$ be a solution to (CP) with initial data u_0, and consider $R(t) = R_\star(t+1)$. The *self-similar change of variables*

$$v(\tau,y) = \frac{R(t)^{d-\gamma}}{\zeta^{d-\gamma}}\,u(t,x) \quad \text{where} \quad \tau = \frac{1}{\sigma}\log\frac{R(t)}{R(0)}\,,\ y = \frac{\zeta\,x}{R(t)}\,, \tag{4.1.15}$$

transforms $u(t,x)$ into a solution to the following nonlinear *Fokker-Planck* type equation

$$\frac{\partial v}{\partial\tau} + |x|^\gamma \mathrm{div}\left(|x|^{-\beta}\,v\,\nabla v^{m-1}\right) = |x|^\gamma \mathrm{div}\left(|x|^{-\beta}\,v\,\nabla|x|^\sigma\right)\,, \tag{4.1.16}$$

with initial data $v_0(y) = \frac{\zeta^{d-\gamma}}{R(0)^{d-\gamma}} u_0(\frac{\zeta\, x}{R(0)})$. Notice that the mass of the initial data is unchanged. Let us recall that the Barenblatt profile $\mathfrak{B}_{M,1}(t,x)$ is tansformed into the stationary profile $\mathcal{B}_M(y)$. In *self-similar* variables the $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ convergence can be restated as

$$\|v(\tau) - \mathcal{B}_M\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \tau \to \infty \,,$$

while the $\mathrm{L}^\infty(\mathbb{R}^d)$ convergence result of (4.1.9) can be restated as

$$\lim_{\tau\to\infty} \|v(\tau) - \mathcal{B}_M\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0\,.$$

4.2 Initial data in $\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$

In this Section we show the results that hold for all data in $\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$, namely, we show that the lower-part of the GHP estimates hold true (Theorem 4.2.1) for (just) locally integrable data: this allows to measure the (infinite) speed of propagation as "fatness of the tails". On the other hand, on the whole space it is not possible to match the lower bounds with similar upper bounds for all initial data in $\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$: we will provide explicit counterexamples and improved lower bounds in Section 4.3. This latter phenomenon, an anomalous tail behaviour, can only happen if we miss a control of the tail of the initial datum: we will show that the sharp tail-condition is encoded in the space $\mathcal{X}$ thoroughly analyzed in Section 4.3. As a consequence of the estimates of this section, we show also uniform convergence in relative error towards equilibrium on compact sets and even on parabolic cones, see Theorem 4.2.4. All of these results are sharp, as shown in Section 4.4.3 by means of suitable counterexamples.

4.2.1 A universal global lower bound: measuring the speed of propagation

We now state the main result of this section, which holds for nonnegative initial data which are merely locally integrable. We recall here a useful quantity, t_* that will appear frequently throughout this section:

$$t_* = t_*(u_0, R) = \kappa_* \, \|u_0\|^{1-m}_{\mathrm{L}^1_\gamma(B_R(0))} \, R^{\frac{1}{\vartheta}}\,. \tag{4.2.1}$$

where $\kappa_* > 0$ depends on d, m, γ, β. This quantity appears in [139, Theorem 3.1], where a complete proof and an explicit expression of κ_* is given. For reader's convenience, we recall the result in Theorem 4.6.3 in Subsection 4.6 in the Appendix.

Theorem 4.2.1. *Let u be a solution to* (CP) *with initial data $0 \le u_0 \in \mathrm{L}^1_{\gamma,\mathrm{loc}}(\mathbb{R}^d)$ and let $t_0, R_0 > 0$ be such that $\|u_0\|_{\mathrm{L}^1_\gamma(B_{R_0}(0))} > 0$. Then there exists $\underline{\tau} > 0$ and $\underline{M} > 0$ such that*

$$u(t,x) \ge \mathfrak{B}(t - \underline{\tau}, x; \underline{M}), \qquad \text{for all } x \in \mathbb{R}^d \text{ and } t \ge t_0\,, \tag{4.2.2}$$

where

$$\underline{\tau} = \frac{1}{2}\,(t_* \wedge t_0) \qquad \text{and} \qquad \underline{M} = b\, \|u_0\|_{\mathrm{L}^1_\gamma(B_{R_0}(0))} \left(1 \wedge \frac{t_0}{t_*}\right)^{\frac{1}{1-m}}\,. \tag{4.2.3}$$

The constant $b > 0$ depends only on d, m, γ, β and has an explicit expression given in the proofs, while t_* is as in (4.2.1).

Measuring the speed of propagation. The above Theorem reveals a remarkable property of solutions to WFDE: the positivity spreads immediately for every nonnegative initial datum, showing infinite speed of propagation. A delicate issue is how to discriminate in a quantitative way among all the possible cases of infinite speed of propagation. Our Theorem shows that we can put a (delayed) fundamental solution as a lower barrier for any data: this is how the WFDE immediately creates a fat tail (inverse power), which is clearly bigger than the "standard Gaussian tail" created by the linear heat equation. This can be expressed as follows:

Corollary 4.2.2 (Minimal tails). *Under the assumption of Theorem 4.2.1 we have that for any $t > 0$*

$$\liminf_{|x|\to\infty} u(t,x)\,|x|^{\frac{\sigma}{1-m}} \geq b_1\, t^{\frac{1}{1-m}} \tag{4.2.4}$$

The constant b_1 depends only on m, d, γ, β and is achieved by the Barenblatt solutions.

We will often call $|x|^{-\sigma/(1-m)}$ *a minimal tail or a Barenblatt tail* . Finding matching upper bounds is simply not possible in such generality, we will need to ask the tail condition (TC) on u_0. We are now going to prove the main result of this section.

Proof of Theorem 4.2.1: Let us explain first the strategy of the proof. The quantities $\underline{\tau}$ and $\underline{M}$ take different forms depending on wether or not $t_* \leq t_0$. We will assume first that $t_0 \geq t_*$, then we will discuss the case $0 < t_0 < t_*$ at the end of the proof.

Let $M_{R_0} = \|u_0\|_{\mathrm{L}^1_\gamma(B_{R_0}(0))}$, $\underline{\tau} = a t_*$ and $\underline{M} = b M_{R_0}$ where $a \in (0,1)$ and $b > 0$ will be explicitly chosen later. Without loss of generality, we prove inequality (4.2.2) only at $t = t_*$, namely

$$u(t_*, x) \geq \mathfrak{B}((1-a)t_*, x; \underline{M})\,. \tag{4.2.5}$$

Once proven at $t = t_*$, the case $t \geq t_*$ will follow by comparison. To prove (4.2.5) we need to determine the values of a, b. We need to separate two cases, namely inside a ball and outside a ball, obtaining different conditions on a, b, respectively conditions (4.2.7) and (4.2.10). Finally we check the compatibility of such conditions and choose a, b explicitly as in (4.2.11).

Condition on a, b inside a ball. We want to find conditions on a, b such that the following inequality holds:

$$\underline{\kappa}_1 \frac{M_{R_0}}{{R_0}^{d-\gamma}} \geq \frac{b^{\sigma\vartheta} M_{R_0}}{b_0^{\frac{1}{1-m}} (1-a)^{(d-\gamma)\vartheta} \kappa_*^{(d-\gamma)\vartheta} {R_0}^{d-\gamma}} = \sup_{x \in B_{2R_0}(0)} \mathfrak{B}(t_* - \underline{\tau}, x; \underline{M})\,, \tag{4.2.6}$$

where $\underline{\kappa}_1$ is as in (4.6.4). It is easily seen that the former is implied by the following condition on a and b:

$$b^{\sigma\vartheta} \leq \kappa_*^{(d-\gamma)\vartheta} \kappa_1 b_0^{\frac{1}{1-m}} (1-a)^{(d-\gamma)\vartheta}\,. \tag{4.2.7}$$

Note that by inequality (4.6.4) the first term in (4.2.6) is bounded above by $\inf_{x\in B_{2R_0}} u(t_*, x)$, therefore we obtain that

$$\inf_{x\in B_{2R_0}} u(t_*, x) \geq \sup_{x \in B_{2R_0}(0)} \mathfrak{B}(t_* - \underline{\tau}, x; \underline{M})\,,$$

inequality (4.2.5) is then proved for any $|x| \leq 2R_0$.

Condition on a, b outside a ball. We want to find suitable conditions on a, b such that (4.2.5) holds in the outer region $|x| > R_0$. Such an inequality will be deduced by applying the comparison on the parabolic boundary of $Q = (\underline{\tau}, t_*) \times B^c_{R_0}(0)$, namely $\partial_p Q = \{\{\underline{\tau}\} \times B^c_{R_0}(0)\} \bigcup \{(\underline{\tau}, t_*) \times \big\{x \in \mathbb{R}^d : |x| = R_0\big\}\}$, see for instance [99, Lemma 3.4].

It is clear that $u(\underline{\tau},x) \geq \mathfrak{B}(0,x;\underline{M}) = \delta_0(x)$, for any $|x| \geq R_0$, hence we just need to prove that

$$u(t,x) \geq \mathfrak{B}(t-\underline{\tau},x;\underline{M}) \quad \text{for any} \quad |x| = R_0,\, t \in (\underline{\tau}, t_*)\,. \tag{4.2.8}$$

The following inequality

$$\underline{\kappa}\left(\frac{a t_*}{{R_0}^{\sigma}}\right)^{\frac{1}{1-m}} \geq \frac{b^{\sigma\vartheta} M_{R_0}^{\sigma\vartheta}}{b_0^{\frac{1}{1-m}}(1-a)^{(d-\gamma)\vartheta} t_*^{(d-\gamma)\vartheta}}\,, \tag{4.2.9}$$

implies that inequality (4.2.8) holds, indeed for any $|x| = R_0$ and $t \in (\underline{\tau}, t_*)$ we have that

$$\begin{aligned} u(t,x) &\geq \inf_{\substack{t\in(a t_*,t_*),\\ x\in B_{2R_0}(0)}} u(t,x) \geq \underline{\kappa}\left(\frac{a t_*}{{R_0}^{\sigma}}\right)^{\frac{1}{1-m}} \geq \frac{b^{\sigma\vartheta} M_{R_0}^{\sigma\vartheta}}{b_0^{\frac{1}{1-m}}(1-a)^{(d-\gamma)\vartheta} t_*^{(d-\gamma)\vartheta}} = \frac{(1-a)^{\frac{1}{1-m}} t_*^{\frac{1}{1-m}}}{\left[b_0 \frac{(1-a)^{\sigma\vartheta} t^{\sigma\vartheta}}{(b M_{R_0})^{\sigma\vartheta(1-m)}}\right]^{\frac{1}{1-m}}} \\ &\geq \frac{(1-a)^{\frac{1}{1-m}} t^{\frac{1}{1-m}}}{\left[b_0 \frac{(1-a)^{\sigma\vartheta} t^{\sigma\vartheta}}{b^{\sigma\vartheta(1-m)} M_{R_0}^{\sigma\vartheta(1-m)}} + b_1 {R_0}^{\sigma}\right]^{\frac{1}{1-m}}} = \sup_{\substack{t\in(a t_*,t_*),\\ |x|=R_0}} \mathfrak{B}(t-\underline{\tau},x;\underline{M})\,. \end{aligned}$$

Recalling that $t_* = \kappa_* M_{R_0}^{1-m} R_0^{1/\vartheta}$ it is easy to show that inequality (4.2.9) is equivalent to the following one

$$b^{\sigma\vartheta} \leq b_0^{\frac{1}{1-m}} \underline{\kappa} \kappa_*^{\frac{1}{1-m}-(d-\gamma)\vartheta} a^{\frac{1}{1-m}} (1-a)^{(d-\gamma)\vartheta}\,, \tag{4.2.10}$$

which is the condition we were looking for.

Compatibility of condition (4.2.7) *and* (4.2.10). Both the conditions are satisfied by the following choice

$$a = \frac{1}{2} \qquad \text{and} \qquad b^{\sigma\vartheta} = \frac{b_0^{\frac{1}{1-m}}}{2^{(d-\gamma)\vartheta}}\left[\underline{\kappa}\kappa_*^{\frac{1}{1-m}-(d-\gamma)\vartheta}\left(\frac{1}{2}\right)^{\frac{1}{1-m}} \wedge \kappa_*^{(d-\gamma)\vartheta}\kappa_1\right]. \tag{4.2.11}$$

This concludes the proof of (4.2.5) in the case when $t_0 \geq t_*$. It only remains to analyze the case when $t_0 < t_*$.

Case $0 < t_0 < t_*$. Without loss of generality, we only need to prove inequality (4.2.2) at time $t = t_0$, the full result will then follow by comparison. Recall the Benilan-Crandall-type estimate, [140],

$$u(t_0,x) \geq u(t_*,x)\left(\frac{t_0}{t_*}\right)^{\frac{1}{1-m}}, \qquad \text{for all } 0 < t_0 < t_*. \tag{4.2.12}$$

Now we recall that inequality (4.2.5) holds under the choices of a, b as in (4.2.11). Using inequality (4.2.5) and inequality (4.2.12) we get

$$\begin{aligned} u(t_0,x) \geq u(t_*,x)\left(\frac{t_0}{t_*}\right)^{\frac{1}{1-m}} &\geq \frac{2^{-\frac{1}{1-m}} t_*^{\frac{1}{1-m}}}{\left[b_0 \frac{2^{-\sigma\vartheta} t_*^{\sigma\vartheta}}{\underline{M}^{\sigma\vartheta(1-m)}} + b_1 |x|^{\sigma}\right]^{\frac{1}{1-m}}}\left(\frac{t_0}{t_*}\right)^{\frac{1}{1-m}} \\ &= \frac{2^{-\frac{1}{1-m}} t_0^{\frac{1}{1-m}}}{\left[b_0 \frac{2^{-\sigma\vartheta} t_0^{\sigma\vartheta}}{\underline{M}^{\sigma\vartheta(1-m)}\left[\left(\frac{t_0}{t_*}\right)^{\frac{1}{1-m}}\right]^{\sigma\vartheta}} + b_1|x|^{\sigma}\right]^{\frac{1}{1-m}}} = \mathfrak{B}\left(t_0 - \frac{t_0}{2}, x; \left(\frac{t_0}{t_*}\right)^{\frac{1}{1-m}}\underline{M}\right). \end{aligned}$$

Recalling that in this case $\underline{\tau} = t_0/2$, the proof is concluded. $\square$

We can now give the proof of Corollary 4.2.2.

Proof of Corollary 4.2.2. Let R_0 be such that $\|u_0\|_{\mathrm{L}^1_\gamma(B_{R_0}(0))} > 0$, $t > 0$, and $0 < \varepsilon < t$. By applying Theorem 4.2.1 at time $t_0 = t - \varepsilon$ and radius R_0 we get the following inequality

$$u(t,x) \geq \mathfrak{B}(t - \underline{\tau}, x; \underline{M}) .$$

As a consequence we obtain

$$\liminf_{|x|\to\infty} u(t,x)\,|x|^{\frac{\sigma}{1-m}} \geq b_1 \left[t - \frac{1}{2}\,(t_* \wedge t_0)\right]^{\frac{1}{1-m}} ,$$

from which (4.2.4) follows just by taking the limit for $\varepsilon \to t$. Notice that in such a limit $t_0 \to 0$. $\square$

4.2.2 Harnack inequality in parabolic cones

We have shown in [139] that nonnegative local solutions to WFDE satisfy Harnack inequalities of various kind: an *elliptic* form (in which the supremum and the infimum are taken at the same time), a *forward* in time (the supremum is taken at a smaller time than the infimum) and a *backward* in time (the supremum is taken at a bigger time than the infimum). We remark that for solutions to the *heat equation* in general only the *forward* Harnack inequality holds. Here we shall prove an *elliptic* form of a Harnack inequality on conical space-time domains of the form

$$K(t) = K_M(t) = \{|x| \leq t^{\vartheta}\, M^{(m-1)\vartheta}\} , \tag{4.2.13}$$

for some fixed $M > 0$. We will call these sets "Parabolic Cones", with a slight abuse of language. Indeed for $\vartheta = 1$, $K(t)$ are really cones in space time domains of the form $\mathbb{R}_+ \times \mathbb{R}^N$. A similar inequality on balls has been proven in [110, Theorem 1.4].

Theorem 4.2.3 (Harnack inequality in parabolic cones)**.** *Let u be a solution to* (CP) *with initial data $0 \leq u_0 \in \mathrm{L}^1_\gamma(\mathbb{R}^d) \setminus \{0\}$. Let $M = \|u_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}$ and $R_0 > 0$ be such that $\|u_0\|_{\mathrm{L}^1_\gamma(B_{R_0}(0))} = M/2$, and let $t_* = \kappa_*\, R_0^{\frac{1}{\vartheta}}\, (M/2)^{\frac{1}{1-m}}$. Then, there exists a positive constant $\mathcal{H}$ such that*

$$\sup_{x\in K(t)} \frac{u(t,x)}{\mathfrak{B}(t,x;M)} \leq \mathcal{H} \inf_{x\in K(t)} \frac{u(t,x)}{\mathfrak{B}(t,x;M)} , \qquad \textit{for any } t \geq 3\,t_*. , \tag{4.2.14}$$

where the constant $\mathcal{H}$ depends only on m, d, γ, β and $K(t)$ depends on M as in (4.2.13).

Proof. By applying Theorem 4.2.1 we deduce that $u(t,x) \geq \mathfrak{B}(t - \underline{\tau}, x; \underline{M})$ with $\underline{\tau} = \frac{t_*}{2} = \frac{\kappa_*}{2}\, R_0^{\frac{1}{\vartheta}}\, \left(\frac{M}{2}\right)^{\frac{1}{1-m}}$ and $\underline{M} = b\,M/2$ where b is as in (4.2.10). In view of the Smoothing Effects (4.6.2) and of inequality (4.2.2), it is enough to prove that there exists $\mathcal{H}$ such that

$$\overline{\kappa}_1\, (b_0 + b_1)^{\frac{1}{1-m}} \leq \mathcal{H}\, b_0^{\frac{1}{1-m}} \frac{t^{(d-\gamma)\vartheta}}{M^{\sigma\vartheta}} \inf_{x\in K(t)} \mathfrak{B}(t - \underline{\tau}, x; \underline{M}) .$$

This amounts to prove that the following quotient is uniformly bounded by $\mathcal{H}$ for $t \geq 3\,t_*$:

$$\overline{\kappa}_1 \left(1 + \frac{b_1}{b_0}\right)^{\frac{1}{1-m}} \frac{M^{\sigma\vartheta}}{t^{(d-\gamma)\vartheta}} \frac{\left[\frac{b_0 (t-\underline{\tau})^{\sigma\vartheta}}{\underline{M}^{2\vartheta(1-m)}} + \frac{b_1 t^{\sigma\vartheta}}{M^{\sigma\vartheta(1-m)}}\right]^{\frac{1}{1-m}}}{(t-\underline{\tau})^{\frac{1}{1-m}}} \leq \mathcal{H} .$$

Since $\underline{\tau} = t_*/2$ we easily conclude that $\mathcal{H}$ can be taken as

$$\mathcal{H} = \overline{\kappa}_1 \left(1 + \frac{b_1}{b_0}\right)^{\frac{1}{1-m}} 5^{\frac{1}{1-m}} \left[b_0 \left(\frac{2}{b}\right)^{\sigma\vartheta} + b_1\right]^{\frac{1}{1-m}} . \ \square$$

4.2.3 Uniform convergence in *relative error* in parabolic cones

In this section we will prove that solutions to (CP) with initial data $u_0 \in \mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$ converge to the Barenblatt profile $\mathfrak{B}(t,x;M)$ in *relative error* uniformly in parabolic cones, and as a consequence uniformly on compact subsets of $\mathbb{R}^d$. To obtain such a result we will use the convergence to the Barenblatt profile in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$, namely

$$\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \to 0 \quad \text{as} \quad t \to \infty\,, \tag{4.2.15}$$

or equivalently, in *self-similar variables*

$$\|v(\tau) - \mathcal{B}_M\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \tau \to \infty\,, \tag{4.2.16}$$

where $v(\tau, y)$ is defined in (4.1.15) and it is a solution to (4.1.16). The proof of (4.2.15) can be done by a straightforward adaptation to our setting of the so called "4 step method", carefully explained in [107, Theorem 1.1]. We leave the details to the interested reader, just noticing that the proof contained in [107] deals with the case $m > 1$, and uses compactly supported initial data, hence compactly supported solutions (when $m > 1$ there is finite speed of propagation). In the present setting, the very same proof works, just by replacing the compactly supported solutions by the ones which satisfy the GHP.

Theorem 4.2.4 (Uniform convergence in *relative error* on parabolic cones)**.** *Assume $m \in \left(\frac{\mathfrak{n}-2}{\mathfrak{n}}, 1\right)$ and let u be a solution to* (CP) *with initial data $0 \le u_0 \in \mathrm{L}^1_\gamma(\mathbb{R}^d) \setminus \{0\}$ and let $M = \|u_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}$. Then for any $\Upsilon > 0$ we have that*

$$\lim_{t\to\infty} \sup_{x \in \{|x| \le \Upsilon\, t^\vartheta\}} \left| \frac{u(t,x) - \mathfrak{B}(t,x;M)}{\mathfrak{B}(t,x;M)} \right| = 0\,. \tag{4.2.17}$$

Remark. As an easy corollary of the previous Theorem, we obtain that

$$\left\| \frac{u(t,x) - \mathfrak{B}(t,x;M)}{\mathfrak{B}(t,x;M)} \right\|_{\mathrm{L}^\infty(K)} \xrightarrow[t\to\infty]{} 0 \qquad \text{for any compact set } K \subset \mathbb{R}^d.$$

This follows from inequality (4.2.17) just by observing that $K \subset \{|x| \le \Upsilon\, t^\vartheta\}$ for some $t_0 > 0$. Before proceeding with the proof, let us define the Hölder seminorm. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $u : \Omega \to \mathbb{R}$ be a function and define for any $\nu \in (0,1)$

$$\lfloor u \rfloor_{C^\nu(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\nu}\,.$$

We say that $u \in C^\nu(\Omega)$ whenever $\lfloor u \rfloor_{C^\nu(\Omega)} < \infty$. We refer to Section 4.7 for more information about the above quantity.

Proof. We split the proof into several steps. First we prove a uniform pointwise estimate on the solution $u(t,x)$ in domains of the form $\{|x| \le CR(t)\}$, where $R(t)$ is as in (4.1.15) and $C > 0$. We remark that for any $t > 0$ we have that $\{|x| \le Ct^\vartheta\} \subset \{|x| \le CR(t)\}$. As a second step we will rescale $u(t,x)$ to self-similar variables (we recall that domains of type $\{|x| \le CR(t)\}$ are transformed into $B_\rho(0)$, where $\rho = \zeta\, C$) and, using the estimates obtained before, we estimate $\lfloor v(\tau,\cdot) - \mathfrak{B}_M \rfloor_{C^\nu(B_{3r})}$ uniformly in time. Finally, by applying a clever interpolation, Lemma 4.7.1 we prove that $\|v(\tau,\cdot) - \mathfrak{B}_M\|_{\mathrm{L}^\infty(B_r)} \to 0$ as $\tau \to \infty$, and finally (4.2.17) follows.

Uniform estimate on $u(t,x)$ in $\{|x|\le 3\,\Upsilon\, R(t)\}$. Let $\rho>0$ be such that $\int_{B_\rho} u_0(x)|x|^{-\gamma}\,\mathrm{d}x=\frac{M}{2}$ and define $t_\star=\kappa_\star\,\rho^{\frac{1}{\vartheta}}\,\left(\frac{M}{2}\right)^{1-m}$ where $\kappa_\star$ is as in (4.6.3). By applying Theorem 4.2.1 and the global smoothing effect, inequality (4.6.2), we obtain that for any $t\ge t_\star$

$$\mathfrak{B}(t-\underline{t},x;\underline{M})\le u(t,x)\le\overline{\kappa}_1\frac{M^{\sigma\vartheta}}{t^{(d-\gamma)\vartheta}}\,,$$

where $\underline{t}=\frac{t_\star}{2}$ and $\underline{M}=\frac{b}{2}\,M$. By the above inequality, we can deduce the following matching lower bound, by means of straightforward estimates relying on the explicit expression of $\mathfrak{B}$: there exists a constant $\underline{\kappa}_1>0$ which depends on $d,m,\gamma,\beta,\Upsilon$ and $\underline{M}$ such that

$$\underline{\kappa}_1\frac{M^{\sigma\vartheta}}{t^{(d-\gamma)\vartheta}}\le u(t,x)\le\overline{\kappa}_1\frac{M^{\sigma\vartheta}}{t^{(d-\gamma)\vartheta}}\quad\text{for any}\quad t\ge t_\star,\ x\in\{|x|\le 3\,\Upsilon R(t)\}\,. \tag{4.2.18}$$

Uniform and Hölder estimates in self-similar variables. We first rescale u in selfsimilar variables, according to (4.1.15), and get $v(\tau,y)$. Analogously, the domain $\{|x|\le 3\,\Upsilon R(t)\}$ is transformed into $B_{3r}(0)$ where $r=\Upsilon\zeta$. Inequality (4.2.18) reads in rescaled variables:

$$\begin{gathered}\frac{\underline{\kappa}_1}{\zeta^{d-\gamma}}\,\vartheta^{\vartheta}M^{\sigma\vartheta}\le v(\tau,y)\le 2\frac{\overline{\kappa}_1}{\zeta^{d-\gamma}}\,\vartheta^{\vartheta}M^{\sigma\vartheta},\\ \text{for any}\quad\tau\ge\frac{1}{\sigma}\log\frac{R(t_\star\vee 1)}{R(0)},\qquad\text{and for any}\quad y\in B_{3r}(0)\,.\end{gathered} \tag{4.2.19}$$

By applying Lemma 4.8.2 we deduce that there exist $\nu>0,\overline{\kappa}>0$ such that for any $\tau\ge\frac{1}{\sigma}\log\frac{R(t_\star\vee1)}{R(0)}+1$ we have that

$$\lfloor v(\tau,\cdot)\rfloor_{C^\nu\left(B_{\frac{3}{2}r}(0)\right)}\le\overline{\kappa}\;\;2\frac{\overline{\kappa}_1}{\zeta^{d-\gamma}}\,\vartheta^{\vartheta}M^{\sigma\vartheta}\,.$$

Using the subadditivity of $\lfloor\cdot\rfloor_{C^\nu(B_r(0))}$ and the fact that the above estimates can also be applied to the Barenblatt profile $\mathcal{B}_M(y)$, we conclude that

$$\lfloor v(\tau,\cdot)-\mathcal{B}_M\rfloor_{C^\nu\left(B_{\frac{3}{2}r}(0)\right)}\le 4\,\overline{\kappa}\,\frac{\overline{\kappa}_1}{\zeta^{d-\gamma}}\,\vartheta^{\vartheta}M^{\sigma\vartheta}\quad\text{for any}\quad\tau\ge\frac{1}{\sigma}\log\frac{R(t_\star\vee 1)}{R(0)}+1\,. \tag{4.2.20}$$

Convergence in L^∞ norm. We only prove the case $0<\gamma<d$, which is the most delicate, the case $\gamma\le 0$ being simpler. In what follows it is convenient to assume that $r\ge 2$, namely that $\Upsilon\ge\frac{2}{\zeta}$, we will overcome this technical assumption at the end of the proof. From the convergence in L^1_γ, formula (4.2.16), we deduce that there exists $\tau_\star$ such that for any $\tau\ge\tau_\star$ we have that $\|v(\tau,\cdot)-\mathcal{B}_M\|_{\mathrm{L}^1_\gamma(B_{\frac{3}{2}r}(0))}\le\frac{|\gamma|}{d}$. We are in the position to apply inequality (4.7.4) of Lemma 4.7.1 to $v(\tau,\cdot)-\mathcal{B}_M$ and get that for any $\tau\ge\tau_\star\vee\frac{1}{\sigma}\log\frac{R(t_\star\vee1)}{R(0)}+1$

$$\begin{aligned}\|v(\tau,\cdot)-\mathcal{B}_M\|_{\mathrm{L}^\infty(B_r(0))}&\le C_{d,\gamma,\nu,p}\,(1+r)^\gamma\,\times\\ &\quad\times\left(1+4\,\overline{\kappa}\,\frac{\overline{\kappa}_1}{\zeta^{d-\gamma}}\,\vartheta^{\vartheta}M^{\sigma\vartheta}\right)^{\frac{d}{d+p\nu}}\|v(\tau,\cdot)-\mathcal{B}_M\|_{\mathrm{L}^1_\gamma(B_{\frac{3}{2}r}(0))}^{\frac{\nu}{d+\nu}}\end{aligned} \tag{4.2.21}$$

where we have used (4.2.20). Since $\mathcal{B}_M(y)\ge(C(M)+r^\sigma)^{\frac{-1}{1-m}}$ on $B_r(0)$, it follows that

$$\sup_{y\in B_r(0)}\left|\frac{v(\tau,y)-\mathcal{B}_M(y)}{\mathcal{B}_M(y)}\right|\le(C(M)+r^\sigma)^{\frac{1}{1-m}}\,\|v(\tau,\cdot)-\mathcal{B}_M\|_{\mathrm{L}^\infty(B_r(0))}\,,$$

which, combined with (4.2.21) and the convergence in L^1_γ, formula (4.2.16), shows that the *relative error* approaches zero as $\tau \to \infty$. Rescaling back, we finally obtain (4.2.17), recalling that $\{|x| \le \Upsilon t^\vartheta\} \subset \{|x| \le \Upsilon R(t)\}$.

It only remains to overcome the technical assumption $\Upsilon \ge \frac{2}{\zeta}$. If $\Upsilon \le \frac{2}{\zeta}$ we can repeat the same argument for $\Upsilon = \frac{2}{\zeta}$. Next, we conclude that (4.2.17) takes place for any $\Upsilon' \le \frac{2}{\zeta}$ using that $\{|x| \le \Upsilon' R(t)\} \subset \{|x| \le \Upsilon R(t)\}$ whenever $\Upsilon' < \Upsilon$. The proof is now concluded. $\square$

4.3 Initial data in $\mathcal{X}$. Global Harnack principle and uniform convergence in relative error

The space $\mathcal{X}$ plays a fundamental role in what follows. As already explained in the introduction, for any $f \in \mathrm{L}^1_\gamma(\mathbb{R}^d)$ the quantity $|f|_{\mathcal{X}}$ defined in (TC) measures the *decay at* ∞ of f. However, to not break the flows of Theorems and ideas we have decided to postpone the analysis of the main feature of $\mathcal{X}$ to Section 4.5.

In what follows we shall prove that $\mathcal{X}$ is the space where the *uniform* convergence in relative error takes place: the GHP will be an instrument of paramount importance in proving this fact.

4.3.1 Upper bound and proof of Theorem 4.1.1

As already observed in the Introduction, Theorem 4.1.1 is divided in two parts: the upper bound and the lower bound of inequality (4.1.7). In this section we are going to discuss the upper bound. The main result of this section is the following Theorem, we postpone the proof of Theorem 4.1.1 to the end of this subsection

Theorem 4.3.1. *Let u be the solution to* (CP) *corresponding to the initial data $0 \le u_0 \in \mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$. Then, for any $t_0 > 0$ there exist $\overline{\tau}, \overline{M} > 0$, explicitly given in* (4.3.9)*, such that*

$$u(t,x) \le \mathfrak{B}(t+\overline{\tau}, x; \overline{M}) \qquad \text{for any } x \in \mathbb{R}^d \text{ and any } t > t_0\,, \tag{4.3.1}$$

if and only if

$$u_0 \text{ satisfies (TC), i.e. } u_0 \in \mathcal{X}\,.$$

The proof of inequality (4.3.1) is constructive and we are able to give values of $\overline{\tau}$ and $\overline{M}$, see formulae (4.3.9) at the end of the proof. Here we just point out that they depend on $M, A, d, m, \gamma, \beta$ and t_0.

Remark 4.3.2. We easily deduce from the above upper bound that

$$\limsup_{|x|\to\infty} u(t,x)\,|x|^{\frac{\sigma}{1-m}} \le b_1\,(t+\overline{\tau})^{\frac{1}{1-m}}\,. \tag{4.3.2}$$

Equality is achieved by the Barenblatt solution translated in time by $\overline{\tau}$. Notice that this maximal tail behaviour only holds for $u_0 \in \mathcal{X}$, in which case it matches the optimal minimal behaviour given in Corollary 4.2.2. These two pieces of information combine well and allow to deduce the sharp behaviour at infinity, see Section 4.3.4, Corollary 4.3.5.

Let us now prove Theorem 4.3.1.

Proof of Theorem 4.3.1: Let us first explain the strategy of the proof. We will prove inequality (4.3.1) only at time t_0, then, by comparison (see for instance [18, Corollary 9]) it will hold for any

$t \ge t_0$. The proof is divided in several steps: first, we estimate the solution $u(t_0, x)$ in two different regions (on $B_{R_1}(0)$ and on $B_{R_1}(0)^c$, with R_1 to be chosen later), then we find conditions on $\overline{\tau}$ and $\overline{M}$ necessary for inequality (4.3.1) to hold. Finally, we show that such conditions can be fulfilled providing an explicit expression of $\overline{\tau}$ and $\overline{M}$ in terms of t_0, M and A.

Estimate inside a ball. We want to find suitable conditions on $\overline{M}, \overline{\tau}$ and R_1 such that

$$u(t_0,x) \le \mathfrak{B}(t_0+\overline{\tau}, x; \overline{M}) = \frac{(t_0+\overline{\tau})^{\frac{1}{1-m}}}{\left[b_0 \frac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}} + b_1|x|^\sigma\right]^{\frac{1}{1-m}}}, \qquad \text{holds for all } |x| \le R_1. \tag{4.3.3}$$

Recall that $M = \int_{\mathbb{R}^d} u_0|x|^{-\gamma}\,\mathrm{d}x$. Inequality (4.6.2) implies that

$$u(t_0,x) \le \overline{\kappa}_1 t_0^{-(d-\gamma)\vartheta} M^{\sigma\vartheta} \qquad \text{for any } x\in\mathbb{R}^d \text{ and } t_0>0.$$

In view of the above inequality, to prove (4.3.3) it is enough to find suitable $\overline{M}, \overline{\tau}$ and R_1 such that

$$\overline{\kappa}_1 \frac{M^{\sigma\vartheta}}{t_0^{(d-\gamma)\vartheta}} \le \frac{(t_0+\overline{\tau})^{\frac{1}{1-m}}}{\left[b_0 \frac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}} + b_1|x|^\sigma\right]^{\frac{1}{1-m}}} \qquad \text{for any } |x| \le R_1.$$

Since the righthand side is decreasing in $|x|$ it suffices to have the previous inequality at $|x| = R_1$, i.e.

$$b_0 \frac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}} + b_1 R_1^\sigma \le \frac{(t_0+\overline{\tau}) t_0^{(d-\gamma)\theta(1-m)}}{\overline{\kappa}_1^{1-m} M^{\sigma\vartheta(1-m)}}. \tag{4.3.4}$$

Inequality (4.3.4) is nothing but a first condition on $\overline{M}, \overline{\tau}$ and R_1 in order to guarantee the validity of (4.3.3).

Estimate outside a ball. The goal of this step is to extend inequality (4.3.3) outside a ball, namely for all $|x| \ge R_1$. This will end up to conditions on $\overline{M}, \overline{\tau}$ and R_1 different from (4.3.4). In the next step we will take care of checking the compatibility of the two conditions.

We first prove that for any fixed $t_0 > 0$ there exists $C_1 = C_1(t_0, A) > 0$ such that

$$u(t_0,x) \le \frac{C_1}{|x|^{\frac{\sigma}{(1-m)}}} \qquad \text{for any } |x| > R_1. \tag{4.3.5}$$

Let $x \in \mathbb{R}^d$, $|x| \ge R_1$ and let R be such that $B_{2R}(x) \subset B_{2R}(0)^c$, for instance $R = |x|/16$. Applying inequality (4.6.1) to $u(t_0, x)$ in the ball $B_R(x)$, we get

$$\begin{aligned}
u(t_0,x) &\le \frac{\overline{\kappa}_1}{t_0^{(d-\gamma)\vartheta}} \left[\int_{B_{2R}(x)} u_0(y)\,|y|^{-\gamma}\,\mathrm{d}y\right]^{\sigma\vartheta} + \overline{\kappa}_2 \left[\frac{t_0}{R^\sigma}\right]^{\frac{1}{1-m}} \\
&\le \frac{\overline{\kappa}_1}{t_0^{(d-\gamma)\vartheta}} \left[\int_{B^c_{2R}(0)} u_0(y)\,|y|^{-\gamma}\,\mathrm{d}y\right]^{\sigma\vartheta} + \overline{\kappa}_2 (16)^{-\frac{\sigma}{1-m}} \left(\frac{t_0}{|x|^\sigma}\right)^{\frac{1}{1-m}} \\
&\le \frac{\overline{\kappa}_1 8^{\frac{\sigma}{1-m}}}{t_0^{(d-\gamma)\vartheta}} \frac{A^{\sigma\vartheta}}{|x|^{\frac{\sigma}{1-m}}} + \frac{\overline{\kappa}_2}{16^{\frac{\sigma}{1-m}}} \left(\frac{t_0}{|x|^\sigma}\right)^{\frac{1}{1-m}} \quad \le \frac{C_1}{|x|^{\frac{\sigma}{1-m}}},
\end{aligned}$$

where in the third line we have used that $\int_{B^c_R(0)} u_0|x|^{-\gamma}\,\mathrm{d}x \le AR^{(d-\gamma)-\frac{2+\beta-\gamma}{1-m}}$ with $R = |x|/16$ and that $C_1 = C_1(t_0, A)$ is given by

$$C_1 = 8^{\frac{\sigma}{1-m}} \frac{\overline{\kappa}_1}{t_0^{(d-\gamma)\vartheta}} A^{\sigma\vartheta} + \frac{\overline{\kappa}_2}{16^{\frac{\sigma}{1-m}}} t_0^{\frac{1}{1-m}}.$$

Hence inequality (4.3.5) holds. It only remains to show that

$$\frac{C_1}{|x|^{\sigma/(1-m)}} \le \frac{(t_0+\overline{\tau})^{\frac{1}{1-m}}}{\left[b_0\frac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}}+b_1|x|^\sigma\right]^{\frac{1}{1-m}}} \qquad \text{for any } |x|\ge R_1. \tag{4.3.6}$$

This will give a condition on $\overline{\tau}, \overline{M}$ and R_1, as we explain next. Indeed, the above inequality is equivalent to

$$b_1 C_1^{1-m} + b_0\frac{(t_0+\overline{\tau})^{\sigma\vartheta}}{|x|^\sigma\overline{M}^{\sigma\vartheta(1-m)}} \le t_0+\overline{\tau}.$$

It is indeed enough to choose $R_1>0$ such that

$$b_1 C_1^{1-m} + b_0\frac{(t_0+\overline{\tau})^{\sigma\vartheta}}{R_1^\sigma\overline{M}^{\sigma\vartheta(1-m)}} \le t_0+\overline{\tau}, \tag{4.3.7}$$

since the second term in left-hand side is decreasing in $|x|$. We conclude that inequality (4.3.3) holds for any $|x|\ge R_1$ whenever $\overline{\tau}, \overline{M}$ and R_1 satisfy condition (4.3.7).

Compatibility among the conditions (4.3.4) and (4.3.7). We only need to show the compatibility of the conditions that imply the main estimates of the previous steps, i.e. that inequality (4.3.3) holds for all $x\in\mathbb{R}^d$. The two conditions (4.3.4) and (4.3.7) correspond to the following system of inequalities

$$\mathbf{(A)} = \begin{cases} b_1 C_1^{1-m} R_1^\sigma + b_0\dfrac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}} \le R_1^\sigma(t_0+\overline{\tau})\,, \\[2ex] b_1 R_1^\sigma + b_0\dfrac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}} \le \dfrac{(t_0+\overline{\tau})t_0^{(d-\gamma)\theta(1-m)}}{\overline{\kappa}_1^{1-m}M^{\sigma\vartheta(1-m)}}\,. \end{cases}$$

It is convenient to simplify the above system in order to be able to make explicit choices of $\overline{\tau}, \overline{M}$ and R_1. The first simplification is the following:

$$\mathbf{(B)} = \begin{cases} b_1\left(1\vee C_1\right)^{1-m} R_1^\sigma \le \dfrac{t_0+\overline{\tau}}{2}\left[R_1^\sigma \wedge \dfrac{t_0^{(d-\gamma)\theta(1-m)}}{\overline{\kappa}_1^{1-m}M^{\sigma\vartheta(1-m)}}\right], \\[2ex] b_0\dfrac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}} \le \dfrac{t_0+\overline{\tau}}{2}\left[R_1^\sigma \wedge \dfrac{t_0^{(d-\gamma)\theta(1-m)}}{\overline{\kappa}_1^{1-m}M^{\sigma\vartheta(1-m)}}\right]. \end{cases} \tag{4.3.8}$$

It is clear that any choice of $\overline{\tau}, \overline{M}$ and R_1 that satisfies **(B)** also satisfies **(A)**. We need a further simplification, but this time we will choose $R_1 = R_1(R_0, t_0, M)$ in a particular way, as follows

$$R_1 := \left(\frac{t_0^{(d-\gamma)\theta}}{\overline{\kappa}_1 M^{\sigma\vartheta}}\right)^{\frac{1-m}{\sigma}} \qquad \text{so that} \quad R_1^\sigma = \frac{t_0^{(d-\gamma)\theta(1-m)}}{\overline{\kappa}_1^{1-m}M^{\sigma\vartheta(1-m)}}\,,$$

and system **(B)** simplifies to

$$\mathbf{(B')} = \begin{cases} b_1\left(1\vee C_1\right)^{1-m} \le \dfrac{(t_0+\overline{\tau})}{2}\,, \\[2ex] b_0\dfrac{(t_0+\overline{\tau})^{\sigma\vartheta}}{\overline{M}^{\sigma\vartheta(1-m)}} \le \dfrac{(t_0+\overline{\tau})}{2}\dfrac{t_0^{(d-\gamma)\theta(1-m)}}{\overline{\kappa}_1^{1-m}M^{\sigma\vartheta(1-m)}}\,, \end{cases}$$

which can be simplified as follows

$$\textbf{(B')} = \begin{cases} \overline{\tau} \geq \left[2b_1\left(1 \vee C_1\right)^{1-m} - t_0\right], \\ \overline{M} \geq \left(2b_0\overline{\kappa}_1^{1-m}\right)^{\frac{1}{\sigma\vartheta(1-m)}} \left(\dfrac{t_0+\overline{\tau}}{t_0}\right)^{\frac{d-\gamma}{\sigma}} M. \end{cases}$$

It is now clear that choosing $\overline{\tau} = \overline{\tau}(t_0, M_\infty, C_1, R_1)$ and $M = M(\overline{\tau}, t_0, M)$ of the form

$$\overline{\tau} := 0 \vee \left[2b_1\left(1 \vee C_1\right)^{1-m} - t_0\right] \quad \text{and} \quad \overline{M} := \left(2b_0\overline{\kappa}_1^{1-m}\right)^{\frac{1}{\sigma\vartheta(1-m)}} \left(\frac{t_0+\overline{\tau}}{t_0}\right)^{\frac{d-\gamma}{\sigma}} M,$$

implies the validity of the two inequalities of system **(B')**, hence of system **(B)**, and finally of **(A)**.

• Values of the constants. Letting $A := |u_0|_{\mathcal{X}}$ and $M := \|u_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}$, we have

$$\begin{aligned} \overline{\tau} &:= 0 \vee \left\{2b_1\left[1 \vee \left(8^{\frac{\sigma}{1-m}}\frac{\overline{\kappa}_1 A^{\sigma\vartheta}}{t_0^{(d-\gamma)\vartheta}} + \frac{\overline{\kappa}_2 t_0^{\frac{1}{1-m}}}{16^{\frac{\sigma}{1-m}}}\right)\right]^{1-m} - t_0\right\}, \\ \overline{M} &:= \left(2b_0\overline{\kappa}_1^{1-m}\right)^{\frac{1}{\sigma\vartheta(1-m)}} \left(\frac{t_0+\overline{\tau}}{t_0}\right)^{\frac{d-\gamma}{\sigma}} M, \end{aligned} \tag{4.3.9}$$

where $\overline{\kappa}_1, \overline{\kappa}_2 > 0$ depend on d, m, γ, β, and they have an explicit expression given at the end of the proof of Theorem 1.2 in [139]. The proof is concluded. □

Proof of Theorem 4.1.1. The proof is a simple combination of Theorem 4.3.1 and Theorem 4.2.1. □

4.3.2 Uniform convergence of the *relative error* for initial data in $\mathcal{X}$

As already mentioned in the introduction, we will prove that *uniform* convergence in relative error holds for data in $\mathcal{X}$. Such kind of convergence is not common in the literature. For the case $\gamma = \beta = 0$ such a result appeared for the first time, to the best of our knowledge, by Vázquez in [107], and it was proven under the additional hypothesis that the initial data decay at infinity as the Barenblatt profile, namely that there exists $R > 0$ such that

$$u_0(x) \leq |x|^{-\frac{2}{1-m}} \qquad \text{for any} \quad |x| \geq R.$$

In [107], Vázquez also asserts that the decay condition may be weakened to an integral one, namely

$$\int_{|y-x|\leq\frac{|x|}{2}} |u_0(y)|\,\mathrm{d}y = \mathcal{O}\left(|x|^{d-\frac{2}{1-m}}\right) \qquad \text{as} \qquad |x| \to \infty,$$

very similar to (TC). In [118], Carrillo and Vázquez computed the *rate of convergence* for radial initial data which satisfy the above decay property, the authors have proven that

$$\left\|\frac{u(t,x) - \mathfrak{B}(t,x;M)}{\mathfrak{B}(t,x;M)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = O(t^{-1}), \tag{4.3.10}$$

for all the range $\frac{d-2}{d} < m < 1$. In particular, again in [118], Carrillo and Vázquez leave unanswered the following question

What is the largest class of initial data for which (4.3.10) *holds?*

Many have contributed to this problem, see for instance [126, 128], more will be discussed in subsection 4.3.5. Here we prove that the *largest class* of initial data for which the *uniform* convergence in relative error takes place is $\mathcal{X}$, providing a partial answer to the open problem posed by Carrillo and Vázquez.

Finally, let us briefly comment on the role of the GHP in what follows. It was already known that convergence in relative error is uniform in parabolic cylinders, see [106] and [107]. To prove that such a convergence takes place uniformly in the whole $\mathbb{R}^d$ one needs a uniform control of the tail of the solution. Here we prove that the GHP provides the right instrument to obtain such a control.

Theorem 4.3.3 (Global Uniform Convergence in *relative error*)**.** *Let $m \in \left(\frac{\mathfrak{n}-2}{\mathfrak{n}}, 1\right)$ and let u be a solution to* (CP) *with initial data $0 \le u_0 \in \mathcal{X} \setminus \{0\}$. Then we have that*

$$\lim_{t\to\infty} \left\| \frac{u(t,x) - \mathfrak{B}(t,x;M)}{\mathfrak{B}(t,x;M)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0\,, \qquad \textit{where} \qquad M = \|u_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}. \tag{4.3.11}$$

Proof. It is convenient to work in self-similar variables: we transform $u(t,x)$ into $v(\tau,y)$ accordingly to formula (4.1.15). We will prove that for any $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that

$$\left\| \frac{v(\tau,y) - \mathcal{B}_M(y)}{\mathcal{B}_M(y)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < 2\varepsilon \qquad \text{for any} \qquad \tau \ge \tau_\varepsilon\,. \tag{4.3.12}$$

We argue that we only need to prove the following claim.
Claim. For any $1 > \varepsilon > 0$ there exists $\rho_\varepsilon >$ and $\overline{\tau}_\varepsilon > 0$ such that

$$\sup_{|y|\ge\rho_\varepsilon} \left| \frac{v(\tau,y) - \mathcal{B}_M(y)}{\mathcal{B}_M(y)} \right| < \varepsilon \quad \text{for any} \quad \tau \ge \overline{\tau}_\varepsilon\,. \tag{4.3.13}$$

Indeed, once the *Claim* is proved we just have to combine it with the convergence inside parabolic cones, i.e. the main result of Theorem 4.2.4 (recall the change of variables (4.1.15) transforms the parabolic cones $\{|x| \le \Upsilon R(t)\}$ into balls of the form $\{|y| \le \Upsilon\}$):

$$\left\| \frac{v(\tau,y)}{\mathcal{B}_M(y)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \left\| \frac{v(\tau,y)}{\mathcal{B}_M(y)} - 1 \right\|_{\mathrm{L}^\infty(\{|y|\le\Upsilon\})} + \left\| \frac{v(\tau,y)}{\mathcal{B}_M(y)} - 1 \right\|_{\mathrm{L}^\infty(\{|y|\ge\Upsilon\})} \le 2\,\varepsilon\,,$$

from which inequality (4.3.12) follows.
Proof of the Claim. Let $t_0, R_0 > 0$ be such that $\|u_0\|_{\mathrm{L}^1_\gamma(B_{R_0}(0))} > 0$. We know by Theorem 4.1.1 that

$$\mathfrak{B}(t - \underline{t}, x; \underline{M}) \le u(t,x) \le \mathfrak{B}(t + \overline{t}, x; \overline{M})\,.$$

for suitable $\overline{t}, \underline{t} > 0$ and $\overline{M}, \underline{M} > 0$. As a consequence, recalling the change of variables (4.1.15), we get

$$a(t)^{d-\gamma}\, (1 \wedge a(t))^{\frac{-\sigma}{1-m}}\, \mathcal{B}_{\underline{M}}(y) \;\le v(\tau,y) \le\; b(t)^{d-\gamma}\, (b(t) \vee 1)^{\frac{-\sigma}{1-m}}\, \mathcal{B}_{\overline{M}}(y)\,. \tag{4.3.14}$$

where $R(t) = R_\star(t+1)$ and

$$\tau = \frac{1}{\sigma} \log \frac{R(t)}{R(0)} \qquad a(t) = \frac{R_\star(t+1)}{R_\star(t+\underline{t})} \quad \text{and} \quad b(t) = \frac{R_\star(t+1)}{R_\star(t+\overline{t})}\,.$$

Since $a(t), b(t) \to 1$ as $t \to \infty$ we deduce that there exists $\tau_\varepsilon > 0$ such that

$$\left(1 - \frac{\varepsilon}{3}\right) \mathcal{B}_{\underline{M}}(y) \le \; v(\tau,y) \; \le \left(1 + \frac{\varepsilon}{3}\right) \mathcal{B}_{\overline{M}}(y) \qquad \text{for every } \tau > \tau_\varepsilon. \tag{4.3.15}$$

Recall that all the Barenblatt solutions $\mathcal{B}_M$ have the same behaviour at infinity, which is independent of the mass M, namely $\lim_{|y|\to\infty} \mathcal{B}_{M_1}(y)/\mathcal{B}_{M_2}(y) = 1$ for any $M_1, M_2 > 0$. Hence, there exists $\rho_\varepsilon = \rho_\varepsilon(\underline{M}, \overline{M}) > 0$ such that

$$1 - \frac{\varepsilon}{3} \le \frac{\mathcal{B}_{\underline{M}}(y)}{\mathcal{B}_M(y)} \quad \text{and} \quad \frac{\mathcal{B}_{\overline{M}}(y)}{\mathcal{B}_M(y)} \le 1 + \frac{\varepsilon}{3}, \qquad \text{for any} \quad |y| \ge \rho_\varepsilon .$$

Combining the above inequality with (4.3.15) we obtain the proof of the *Claim*. The proof is concluded. $\square$

4.3.3 Proof of Theorem 4.1.3

We finally provides in this section the proof of Theorem 4.1.3. We recall here that Theorem 4.1.3 provide a characterization of the convergence in *uniform* relative error towards the Barenblatt profile, i.e. the limit

$$\lim_{t\to\infty} \left\| \frac{u(t,x)}{\mathfrak{B}(t,x;M)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0 , \qquad \text{where} \qquad M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)},$$

holds *if and only if* the initial data $u_0 \in \mathcal{X}$. Here we want again to stress two main points.

First, the validity of the above limit was known in the literature (see [107, 118]) only under stronger assumptions on the initial data, namely that the initial data $u_0 \lesssim |x|^{-\frac{2}{1-m}}$ for $|x| \ge R$ and R big enough. It other words, the initial data resembles the Barenblatt profile "at infinity". Here we clarify that this hypothesis is sufficient but not necessary. As we will see in Section 4.5, $\mathcal{X}$ contains (even radial) functions which do not satisfy the above condition.

Second, if uniform convergence takes place, then the initial data u_0 belong to $\mathcal{X}$. Therefore, the condition $u_0 \in \mathcal{X}$ represents a necessary condition for such a convergence to hold.

Proof of Theorem 4.1.3. We notice first that the validity of (4.1.11) under the assumption (TC) has already been proven in Theorem 4.3.3 so we need only to prove the converse. The proof will be based on the following claim.

Claim. Let $m \in \left(\frac{\mathrm{n}-2}{\mathrm{n}}, 1\right)$ and let u be a solution to (CP) with initial data $0 \le u_0 \in \mathrm{L}^1_\gamma(\mathbb{R}^d)$, then for any $R > 0$ and for any $t, s \ge 0$ there exist constants $C_1, C_2 > 0$ which depend on m, d, γ, β such that

$$\int_{B^c_{2R}(0)} u(t,x)|x|^{-\gamma}\, \mathrm{d}x \le C_1 \int_{B^c_R(0)} u(s,x)|x|^{-\gamma}\, \mathrm{d}x + C_2\, |t-s|^{\frac{1}{1-m}}\, R^{(d-\gamma)-\frac{\sigma}{1-m}} . \tag{4.3.16}$$

Inequality (4.3.16) resembles inquality 3.1 of [99, Lemma 3.1]. To prove (4.3.16) one can adapt the techniques used in Lemma 1.2.1 of Part I, Chapter 1: we only need to change the test function from a smooth, compactly supported function ϕ to $\psi = 1 - \phi$ and integrate on the complementary of $B_R(0)$ instead of taking the integrals in $B_R(0)$. Since the solutions are in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ integrating on the complementary of a ball does not represent a problem. We do not give a proof of this *Claim* since the proof is very similar to the one of [139, Proposition 2,4]. Let us proceed with the rest of the proof.

Assume now that (4.1.11) holds, we deduce that there exists a time $\bar{t} > 0$ such that for any $x \in \mathbb{R}^d$

$$\left| \frac{u(\bar{t},x) - \mathfrak{B}(\bar{t},x;M)}{\mathfrak{B}(\bar{t},x;M)} \right| < 1 ,$$

by the triangle inequality we conclude from the above that

$$u(\bar{t},x) \le 2\,\mathfrak{B}(\bar{t},x;M)\,.$$

A simple computation then shows that for any $R>0$ there exists a constant $\kappa>0$ which depend on m,d,γ,β and on $\bar{t}$ such that

$$\int_{B_R^c(0)} u(\bar{t},x)|x|^{-\gamma}\,\mathrm{d}x \le \kappa\, R^{d-\frac{\sigma}{1-m}}\,.$$

By applying (4.3.16) at time $t=0$ and $s=\bar{t}$ we conclude that for any $R>0$

$$(2R)^{\frac{\sigma}{1-m}-d}\int_{B_{2R}^c(0)} u_0(x)|x|^{-\gamma}\,\mathrm{d}x \le C\,\kappa^{\frac{1}{1-m}} + C_{m,d,\gamma,\beta}^{\frac{1}{1-m}}\,|\bar{t}|^{\frac{1}{1-m}}\,,$$

which shows that the initial data u_0 satisfies the tail condition (TC). The proof is then concluded.

4.3.4 Harnack inequalities for quotients and sharp behaviour at infinity

Here we prove a Harnack inequality which, to the best of our knowledge, was not known in the context of the FDE, even though similar results were proven in [110]. The Harnack inequality we present here is global, i.e. it holds on the whole $\mathbb{R}^d$, and for initial data $u_0\in\mathcal{X}$.

Theorem 4.3.4. *Let u be a solution to* (CP) *with $0\le u_0\in\mathcal{X}\setminus\{0\}$ such that $\|u_0\|_{m,\gamma,\beta}=A$ and $\|u_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}=M$ and let $R_0>0$ be such that $\|u_0\|_{\mathrm{L}^1_\gamma(B_{R_0}(0))}=M/2$. Then there exists a constant* H, *which depends only on m,d,γ,β, such that*

$$\sup_{x\in\mathbb{R}^d}\frac{u(t,x)}{\mathfrak{B}(t,x;M)} \le \mathrm{H}\,\inf_{x\in\mathbb{R}^d}\frac{u(t,x)}{\mathfrak{B}(t,x;M)} \quad \text{for any} \quad t\ge\bar{t}$$

where

$$\bar{t} = 3\,\max\left\{A^{1-m}\left(\frac{\overline{\kappa}_1}{\overline{\kappa}_2}\right)^{\frac{1-m}{\sigma\vartheta}}2^{\frac{7}{\vartheta}},\kappa_*\,R_0^{\frac{1}{\vartheta}}\,(M/2)^{\frac{1}{1-m}}\right\}.$$

The constants $\overline{\kappa}_1,\overline{\kappa}_2$ and κ_* are as in (4.6.1) *and as in* (4.6.3) *respectively.*

Proof. In what follows we shall assume without loss of generality that

$$\kappa_2^{\frac{1}{1-m}} \ge b_1^{-1}\,2^{4\sigma+m-2}\,. \tag{4.3.17}$$

Indeed, since κ_2 comes from the upper bound (4.6.1) we can choose it as large as needed. By applying Theorem 4.3.1 at time $t_0=A^{1-m}\left(\frac{\overline{\kappa}_1}{\overline{\kappa}_2}\right)^{\frac{1-m}{\sigma\vartheta}}2^{\frac{7}{\vartheta}}$ and Theorem 4.2.1 at time $t_1=\kappa_\star\,R_0^{\frac{1}{\vartheta}}\,(M/2)^{\frac{1}{1-m}}$ we obtain that for any $t\ge\bar{t}$ the following inequality holds

$$\mathfrak{B}(t-\underline{\tau},x;\underline{M}) \le u(t,x) \le \mathfrak{B}(t+\overline{\tau},x;\overline{M})\,, \tag{4.3.18}$$

where

$$\underline{\tau}=t_1/2=\left(\frac{\kappa_\star}{2}\right)R_0^{\frac{1}{\vartheta}}\left(\frac{M}{2}\right)^{\frac{1}{1-m}}\,,\qquad \underline{M}=b\,\frac{M}{2}\,,$$

and

$$\overline{\tau} = (b_1\, 2^{2-m-4\sigma}\, (\overline{\kappa}_2)^{1-m} - 1)t_0\,, \qquad \overline{M} = (2b_0\overline{\kappa}_1)^{1-m}\left(b_1\, 2^{2-m-4\sigma}\, (\overline{\kappa}_2)^{1-m}\right)^{\frac{d-\gamma}{\sigma}} M\,.$$

Here is the point where the assumption (4.3.17) enters the game, since it implies that $\overline{\tau} \geq 0$. By inequality (4.3.18) it is enough to show that there exists a constant H such that for any $t \geq \overline{t}$

$$\sup_{x\in\mathbb{R}^d} \frac{\mathfrak{B}(t+\overline{\tau}, x; \overline{M})}{\mathfrak{B}(t, x; M)} \leq \mathrm{H} \inf_{x\in\mathbb{R}^d} \frac{\mathfrak{B}(t-\underline{\tau}, x; \underline{M})}{\mathfrak{B}(t, x; M)}\,.$$

A simple computation, which is left to the interested reader, shows that the previous inequality holds with a constant which depends only on m, d, γ, β and not on the mass M neither on the parameter A. The proof is then concluded. □

One can interpret the above Harnack inequality as a control of the tail of the solution. Indeed, for solutions with initial data in $\mathcal{X}$ we have the following Corollary which describes the behaviour at "infinity" of such solutions.

Corollary 4.3.5 (Sharp behaviour at infinity)**.** *Let u be the solution to* (CP) *corresponding to the initial data $0 \leq u_0 \in \mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$. Then we have that*

$$1 \leq \frac{\limsup_{|x|\to\infty} u(t,x)\,|x|^{\frac{\sigma}{1-m}}}{\liminf_{|x|\to\infty} u(t,x)\,|x|^{\frac{\sigma}{1-m}}} \leq \left(1 + \frac{\overline{\tau}}{t}\right)^{\frac{1}{1-m}} \qquad \textit{if and only if} \qquad u_0 \in \mathcal{X}\setminus\{0\}\,.$$

Here, $\overline{\tau}$ depends on the initial data and is as in Theorem 4.3.1.

Proof. We just have to combine inequality (4.2.4) of Corollary 4.2.2 with inequality (4.3.2) of Remark 4.3.2. □

The above inequality gives us some interesting information about the sharp behaviour at infinity of solutions to (CP) when the initial data $u_0 \in \mathcal{X}$. It is remarkable that the behaviour at infinity does not depend on the mass: more precisely, as an easy consequence of the above Corollary, we can show that for all $\overline{M}, \underline{M} > 0$, there exists $\overline{\tau} > 0$ such that

$$1 \leq \frac{\limsup_{|x|\to\infty} u(t,x)\,\mathfrak{B}^{-1}(t,x;\overline{M})}{\liminf_{|x|\to\infty} u(t,x)\;\mathfrak{B}^{-1}(t,x;\underline{M})} \leq \left(1 + \frac{\overline{\tau}}{t}\right)^{\frac{1}{1-m}} \qquad \text{if and only if} \qquad u_0 \in \mathcal{X}\setminus\{0\}\,.$$

The above inequalities are sharp, indeed equality is attained by Barenblatt profiles, possibly with different mass.

4.3.5 Almost universal rates of convergence in $\mathcal{X}$

As we have mentioned in the Introduction of Part II, we know that solutions starting from $0 \leq u_0 \in \mathcal{X}$, will eventually converge to a Barenblatt profile $\mathfrak{B}_M$ (with the same mass as u_0), i.e. an element of the manifold $\mathcal{B}$. The natural question that we address here is: are there "universal rates" of convergence towards $\mathcal{B}$? More precisely:

in self-similar variables, can we find a speed of convergence to the stationary profile which is valid for all solutions starting from data in $\mathcal{X}$?

The answer to this question is delicate and can not be easily given for all $m \in (0,1)$, neither for all $m \in (\frac{\mathrm{n}-2}{\mathrm{n}}, 1)$. Some preliminary remarks are in order. In the case $\gamma = \beta = 0$ the question has a long history (see [119]): for $\frac{d-2}{d} < m < 1$ under suitable assumptions, it has been proven in [108, 116, 120, 12, 121, 117, 122], with techniques that involve the relative entropy functional (introduced in [123, 124]) or exploiting the so called Bakry-Émery methods, (see [125]), that there exist (sharp) rates of convergence in different topologies, the most common being the d_1 (see Introduction to Part II) . The rate t^{-1} of convergence in uniform relative error has been computed first in [118] for radial data in the whole range $\frac{d-2}{d} < m < 1$, later in [126] such a rate was extended to a larger class of data. In a quite long series of papers, [127, 128, 129, 53] similar results were obtained in the whole range $0 < m < 1$, we recall that in the range $0 < m < \frac{d-2}{d}$ there is a dramatic change in the behaviour of solutions since they may vanish in finite time, see [48, 4]. In the general case $\gamma \neq 0, \beta \neq 0$, rates of convergence were obtained in [18, 19].

In what follows we will show how we can combine the techniques of this paper with the ones used in [128, 53, 129], to obtain rates of convergence to the Barenblatt profile with an (almost) uniform rate in the whole $\mathcal{X}$. For reasons that are not entirely clear up to now, we need to restrict ourselves to the range $\frac{d-1}{d} = m_1 < m < 1$ in the case $\gamma = \beta = 0$, and to the range $\frac{2d-2-\beta-\gamma}{2(d-\gamma)} < m < 1$ for the general case, see [18, 19] for further remarks. The latter restriction is somehow natural, since, at least when $\gamma = \beta = 0$, we have that the FDE is a gradient flow of a displacement convex functional (the relative entropy) with respect to the so-called Wesserstein distance, see [114, 117, 115]. The displacement convexity is lost below m_1. For the sake of completeness we report here our main result, already contained on page 106 reads:

Theorem (**4.1.4** of page 106, Almost sharp universal rates of convergence in the non-weighted case). *Let u be the solution to* (CP) *corresponding to the initial data $0 \le u_0 \in \mathcal{X} \setminus \{0\}$, $\int_{\mathbb{R}^d} u_0 \,\mathrm{d}x = M$, $\int_{\mathbb{R}^d} x u_0(x) \,\mathrm{d}x = 0$ and assume that $\beta = \gamma = 0$. Then, for every $\delta \in (0,1)$ there exist $t_\delta, c_\delta > 0$ (that may also depend on u_0) such that for all $t > t_\delta$*

$$\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}} \qquad \text{and} \qquad t^{d\vartheta}\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}}. \tag{4.3.19}$$

Remark. Notice that the above result is new for the whole space $\mathcal{X}$ even if we are dealing with the case $\gamma = \beta = 0$. Indeed, all the previous results deal with more restrictive assumption as radial data, a very precise control for $|x| \to \infty$ or being sandwiched between two Barenblatt profiles.

When dealing with CKN-weights, the result is a bit weaker, because of the possible lack of C^k regularity at the origin and reads:

Theorem (**4.1.6** of page 106, Almost sharp universal rates of convergence in the weighted case). *Let u be the solution to* (CP) *corresponding to the initial data $0 \le u_0 \in \mathcal{X} \setminus \{0\}$ with $\int_{\mathbb{R}^d} u_0 |x|^{-\gamma} \,\mathrm{d}x = M$ and assume $\gamma < 0$. Then, there exists a $\delta_* \in (0,1)$ such that for every $\delta \in (0,\delta_*)$ there exist $t_\delta, c_\delta > 0$ (that may also depend on u_0) such that for all $t > t_\delta$*

$$\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}} \qquad \text{and} \qquad t^{(d-\gamma)\vartheta}\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_\delta}{t^{1-\delta}}.$$

If we consider radial initial data in $\mathcal{X}$ we can provide a *universal* rate of convergence, very much in the spirit of [118] or [126].

Theorem (**4.1.7** of page 106, Sharp universal rates for radial data). *Assume $\gamma = \beta = 0$ and let $m \in \left(\frac{d-2}{d}, 1\right)$. Let u be the solution to* (CP) *corresponding to the radial initial data $0 \le u_0 \in \mathcal{X} \setminus \{0\}$,* *with* $\int_{\mathbb{R}^d} u_0 \,\mathrm{d}x = M$. *Then, there exist* $t_0, c_0 > 0$ *(that may also depend on* u_0*) such that for all* $t > t_0$

$$\left\| \frac{u(t)}{\mathfrak{B}(t;M)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_0}{t}. \tag{4.3.20}$$

Remark 4.3.6. As an immediate consequence of (4.3.20) we obtain that for all $t \ge t_0$

$$\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \le \frac{c_0}{t} \qquad \text{and} \qquad t^{(d-\gamma)\vartheta}\|u(t) - \mathfrak{B}(t;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{c_0}{t}.$$

The above Theorem solves a problem left open in [118], i.e. identifying the largest class of nonnegative *radial* L^1 data for which the above rate of convergence holds. Such rates are proven to be sharp, since they are fulfilled by two time-shifted Barenblatt, with the same mass, see [118, 126]. Finally, we observe that, even if we restrict the analysis to radial data, the class $\mathcal{X}$ is much larger than those considered up to know in the literature: we refer to Section 4.5 for examples of functions in $\mathcal{X}$ which do not resemble the behaviour of the Barenblatt profile.

Finally, let us give the proof of the above statements.

Proof of Theorem 4.1.4. Here we exploit the techniques introduced in [128, 53, 129]. Let us rescale $u(t,x)$ to $v(\tau,y)$ according to the change of variables (4.1.15) and define $w = \frac{v(\tau,y)}{\mathcal{B}_M(y)}$ where $M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}$. Let us define the *Free Energy* or *Relative Entropy* $\mathcal{F}[w]$ and the *Fisher Information* $\mathcal{I}[w]$ as

$$\begin{aligned} \mathcal{F}[w(\tau)] = \frac{m}{m-1}\int_{\mathbb{R}^d}\left[\frac{w^m-1}{m} - (w-1)\right]\mathcal{B}_M^m \,\mathrm{d}y \quad \text{and} \\ \mathcal{I}[w] := \frac{m}{1-m}\int_{\mathbb{R}^d} w\,\mathcal{B}_M \left|\nabla\left[\left(w^{m-1}-1\right)\mathcal{B}_M^{m-1}\right]\right|^2 \mathrm{d}y\,. \end{aligned} \tag{4.3.21}$$

It is well known in the literature (see [138, 137, 17, 118, 117]) that the relative entropy controls the L^1 distance between the solution $v(\tau,y)$ and the Barenblatt profile $\mathcal{B}_M$, indeed, by the Csiszár-Kullback inequality one obtains

$$\|v(\tau) - \mathcal{B}_M\|_{\mathrm{L}^1(\mathbb{R}^d)} \le \left(\frac{8}{m}\|\mathcal{B}_M^{2-m}\|_{\mathrm{L}^1(\mathbb{R}^d)}\right)^{\frac{1}{2}} \sqrt{\mathcal{F}[w]}\,. \tag{4.3.22}$$

therefore a decay of the relative entropy corresponds to a decay of $\|u(t) - \mathcal{B}(t)\|_{\mathrm{L}^1(\mathbb{R}^d)}$. If $v(\tau,y)$ is a solution to (4.1.16) the Fisher information is related to the relative entropy as follows

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}[w] = -\mathcal{F}[w]\,. \tag{4.3.23}$$

Also, notice that for any $m \in \left(\frac{d-1}{d}, 1\right)$ the *Entropy-Entropy Production* inequality

$$4\,\mathcal{F}[w] \le \mathcal{I}[w]\,, \tag{4.3.24}$$

is equivalent to the optimal Gagliardo-Nirenberg inequality, see [12]. Recall that the best constant in the Gagliardo-Nirenberg inequality is achieved precisely by the Barenblatt profiles. The best constant in (4.3.24) is 4 and it determines, as follows from (4.3.23), the decay of the relative entropy along the flow, namely

$$\mathcal{F}[w] \le \mathcal{F}[w_0]\,e^{-4\tau}\,.$$

The strategy in [128, 53, 129] consists in proving a faster decay of the entropy along the flow using an improved (with a larger constant) entropy-entropy production inequality along the flow. This is done by comparing the relative entropy and the Fisher information with suitably linearized quantities which satisfy a Hardy-Poincaré inequality (with an improved constant). Combining Lemma 3, Theorem 7 of [128] with Lemma 1 of [129] with no difficulties one can prove the following claim.

Claim. For any $0 < \delta < 4\frac{1-\vartheta}{\vartheta}$ there exists a time $\tau_\delta > 0$ such that

$$\left(\frac{4}{\vartheta} - \delta\right) \mathcal{F}[w(\tau)] \le \mathcal{I}[w(\tau)]\,, \qquad \text{for any } \tau \ge \tau_\delta \tag{4.3.25}$$

where ϑ is as in (4.1.2).

Sketch of the proof of the claim. We shall not provide the lengthy details of the proof of the above claim, we will just explain how to deduce it as a straightforward combination of already published results, adapting them to the current notations. The claim follows by formula (11) of [129], that in the current notations takes the form (at least for sufficiently large times)

$$\frac{2\left[\Lambda_{\alpha,d} - d(1-m)\left((1+\varepsilon)^{4(2-m)} - 1\right)\right]}{(1+\varepsilon)^{7-3m}}\mathcal{F}[w(\tau)] \le \mathcal{I}[w(\tau)]\,, \tag{4.3.26}$$

where ε is (roughly speaking) the size of the relative error $|w-1| \sim \varepsilon$, which we need to be small in order to guarantee the validity of the result (note that in formula (11) of [129] $h \sim 1+\varepsilon$). Notice that everything is quantified explicitly in terms of ε in the paper [129] which also relies on precise results of [128, 24]. The smallness of ε for sufficiently large times follows by our Theorem 4.1.1, Global Harnack Principle, together with the uniform convergence in relative norm, Theorem 4.3.3. Recalling now Lemma 1 of [129], we get the expression for $\Lambda_{\alpha,d} = 4\alpha - 2d$, which in our notations becomes $\Lambda_{\alpha,d} = \frac{2}{\vartheta}$. Note that we need to assume that the first moment is fixed, but this is well-known to be true along the nonlinear flow as well, see [129]. This concludes the proof of the claim. For a more complete account of the details of the proof see also Proposition 6.3.1 of chapter 6.

As a consequence of inequality (4.3.25), we obtain a faster decay of the relative entropy and conclude that

$$\|v(\tau) - \mathcal{B}_M\|_{\mathrm{L}^1(\mathbb{R}^d)} \le C\, e^{-\left(\frac{2}{\vartheta}-\frac{\delta}{2}\right)\tau}\,. \tag{4.3.27}$$

By re-scaling back to original variables and observing that $e^{2\tau} = R(t) \sim t^\vartheta$ one concludes that

$$\|u(t,\cdot) - \mathfrak{B}(t;M)\|_{\mathrm{L}^1(\mathbb{R}^d)} \le C\, t^{-1+\frac{\delta}{4}\vartheta}\,,$$

since $\delta > 0$ was arbitrary small we conclude that the left inequality in (4.3.19) holds.

It only remains to prove the second inequality in (4.3.19), to do so we need to invoke the following interpolation Lemma which goes back to Gagliardo (see [141]) and Nirenberg (see [142, Pag. 126]): let $f \in C^k(\mathbb{R}^d) \cap \mathrm{L}^1(\mathbb{R}^d)$ for some $p \ge 1$ and k a positive integer, then

$$\|f\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le C_{p,k,d}\, \|f\|_{C^k(\mathbb{R}^d)}^{\frac{d}{d+k}}\, \|f\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{k}{d+k}}\,, \tag{4.3.28}$$

where $\|\|_{C^k(\mathbb{R}^d)}$ is given by

$$\|f\|_{C^k(\mathbb{R}^d)} := \max_{|\eta|=k}\, \sup_{z\in\mathbb{R}^d} \left|\delta^\eta f(z)\right|,$$

where $|\eta| = \eta_1 + \ldots + \eta_d$ is the length of the multi-index $\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{Z}^d$. We recall that in the case $\gamma = \beta = 0$ solution to (CP) are $C^\infty(\mathbb{R}^d)$ and for any $k \ge 1$ we have that

$$\sup_{\tau\ge\tau_0} \|v(\tau) - \mathcal{B}_M\|_{C^j(\mathbb{R}^d)} < \infty\,,$$

for a proof of the above inequality see [128, Theorem 2 and Theorem 4]. Fix $k \geq 1$ to be chosen later, combining the above interpolation inequality (4.3.28) with the decay of the L^1 norm given in (4.3.27) one obtains

$$\|v(\tau) - \mathcal{B}_M\|_{\mathrm{L}^\infty} \lesssim e^{-\left(\frac{2}{\vartheta}-\frac{\delta}{2}\right)\left(\frac{k}{d+k}\right)},$$

and rescaling back to original variables we easily find that

$$t^{d\vartheta}\|u(t,\cdot) - \mathfrak{B}(t;M)\|_{\mathrm{L}^\infty} \leq C\, t^{-\left(1-\frac{\delta\vartheta}{4}-\frac{d}{d+k}+\frac{d\,\delta\,\vartheta}{4(d+k)}\right)},$$

since both k and δ we arbitrary we conclude that the second inequality in (4.3.19) holds. This concludes the proof. □

Proof of Theorem 4.1.6. The proof is very similar to the one of Theorem 4.1.4, here we only explain the main differences. We cannot reach the rate $t^{-1+\delta}$ for two reasons. The first: we can obtain an inequality as the one (4.3.25), however the constant is smaller that $\left(\frac{4}{\vartheta}-\delta\right)$, see [18, 19]. The second: we need to assume $\gamma < 0$ to obtain an inequality similar to (4.3.28), see Lemma 4.7.1. Lastly, solutions to (CP) do not enjoy C^∞ regularity, indeed the maybe only C^α at the origin (see Part I), so we cannot use inequality (4.3.28) as done in the proof of Theorem 4.1.4. This concludes our considerations. □

Proof of Theorem 4.1.7 Carrillo and Vázquez have proved Theorem II under the assumption that the initial data u_0 is bounded, radially symmetric and satisfies $u_0 = O(|x|^{-\frac{2}{1-m}})$. It is only needed to show that radial data in $\mathcal{X}$ produce solutions that satisfy the decay assumption above for any time $t > t_0$ for some given t_0. This is exactly the statement of the GHP. The proof is concluded. □

4.4 Counterexamples and generalized global Harnack principle

In Section 4.3 we have seen how under some suitable assumption on the decay of the initial data solutions to (CP) have a common behaviour at infinity and for any time $t > 0$ are sandwiched between two Barenblatt profiles, see Theorem 4.1.1. Here we provide an example of solution which does not share this behaviour. For the sake of clarity here we consider solutions to (CP) without weights (i.e. $\gamma = \beta = 0$), in the subsections below we will provide results in the full generality.

Let $m > \frac{d}{d+2}$, $\gamma = \beta = 0$ and consider the solution $w(t,x)$ to (CP) with initial data w_0 given by

$$w_0 = \frac{1}{(1+|x|^2)^{\frac{m}{1-m}}}.$$

It is clear that for w_0 does not satisfy the assumption of Theorem 4.3.1 and, for $|x|$ large enough, we have that $w_0(x) > \mathfrak{B}(t_0,x;M)$ for any $t_0, M > 0$. However, $w_0 \in \mathrm{L}^1(\mathbb{R}^d)$ for any $m > \frac{d}{d+2}$. As will be clear below the behaviour of $w(t,x)$ does not resemble the one given by the Barenblatt profile. Indeed, by constructing explicit sub/super-solutions, we are able to show that for any time $t > 0$

$$\frac{1}{\left((4m(1-m)d\,t+1)^{\frac{1}{1-m}} + |x|^2\right)^{\frac{m}{1-m}}} \leq w(t,x) \leq \frac{(1+t)^{\frac{m}{1-m}}}{(1+t+|x|^2)^{\frac{m}{1-m}}}.$$

The inequality above gives us remarkable insights about the long time behaviour of the solution $w(t,x)$. First, for any time $t > 0$, $w(t,x)$ has a power-like behaviour at infinity, namely $w(t,x) \sim |x|^{\frac{-2m}{1-m}}$ as $|x| \to \infty$, which differs substantially from the Barenblatt's one. Second, due to this anomalous tail behaviour, an inequality as (4.3.1) is simply not possible, due to the not-matching powers of the tail of $w(t,x)$ with respect to the Barenblatt. Third, the existence of the function $w(t,x)$ shows that the *tail-condition* (TC) is not only sufficient but also necessary for Theorem 4.1.1 and 4.3.1 to hold. Finally, the function $w(t,x)$ provides an example of solution which does not converge in *relative-error* to the Barenblatt profile. Indeed, due to the non-matching tail powers we have that for any $t>0$

$$\sup_{x\in\mathbb{R}^d}\Big\|\frac{w(t,x)}{\mathfrak{B}(t,x;M)}-1\Big\|_{\mathrm{L}^\infty}=\infty\,,$$

where $\mathfrak{B}(t,x;M)$ has the same mass of $w(t,x)$. The same considerations apply if we substitute $\mathfrak{B}(t,x;M)$ by any other Barenblatt profile with a different mass.

Such an anomalous behaviour is not peculiar to only $w(t,x)$ and is indeed shared by many other solutions. In what follows we construct a family of sub and super solutions which can be classified with respect to their power-like behaviour at infinity. The construction holds for any γ,β in our range of parameters and for any $m\in(\frac{\mathfrak{n}-2}{\mathfrak{n}},1)$. Therefore all the considerations above will *de facto* apply to solutions to (CP) with general γ and β.

4.4.1 Construction of a subsolution

In the following Proposition we construct an explicit family of sub-solutions which can be classified by the power-like decay at infinity. Every subsolution decays in space slowly than the Barenblatt profile.

Proposition 4.4.1 (Family of Subsolutions). *Let $m\in(\frac{\mathfrak{n}-2}{\mathfrak{n}},1)$, $\varepsilon\in(0,\frac{2}{1-m}-\mathfrak{n})$, $A,B>0$ and $\alpha=\frac{1}{1-m}-\frac{\varepsilon}{2}>0$. Define for some $t_0\in\mathbb{R}$ the function*

$$D(t):=\left(\sigma\,A^{m-1}\,m\,B\,(d-\gamma)\,(1-\alpha(1-m))\,t+t_0\right)^{\frac{1}{1-\alpha(1-m)}}\,. \tag{4.4.1}$$

Then the function

$$\underline{V}(t,x)=\frac{A}{(D(t)+B|x|^\sigma)^\alpha}\in\mathrm{L}^1_\gamma(\mathbb{R}^d) \tag{4.4.2}$$

satisfies, in a suitable sense,

$$\partial_t\underline{V}(t,x)\le|x|^\gamma\mathrm{div}\left(|x|^{-\beta}\nabla\underline{V}^m\right)\,. \tag{4.4.3}$$

If $m\in(\frac{\mathfrak{n}}{\mathfrak{n}+2},1)$ and $\varepsilon\in(0,\frac{2}{1-m}-\mathfrak{n}-2)$ then $|x|^\sigma\,\underline{V}(t,x)\in\mathrm{L}^1_\gamma(\mathbb{R}^d)$.

Remark 4.4.2. We notice that $\|V(t,\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\asymp t^{-\frac{\alpha}{1-\alpha(1-m)}}$ as $t\to\infty$. This is not in contrast with the smoothing effect (inequality (4.6.2)) which implies that any solution $u(t,x)$ to (CP) decays in time as $\|u(t,\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\asymp t^{-(d-\gamma)\vartheta}$. Indeed, a simple computation shows that the condition $\varepsilon\in\left(0,\frac{2}{1-m}-\mathfrak{n}\right)$ implies that $t^{-\frac{\alpha}{1-\alpha(1-m)}}<t^{-(d-\gamma)\vartheta}$. However, as $|x|\to\infty$, $V(t,x)$ exhibits quite an interesting behaviour: $V(t,x)\asymp|x|^{-\sigma\alpha}$. The power $-\sigma\alpha$ do not match the one of the fundamental solution, indeed, for any choice of the mass M, as $|x|\to\infty$ we have $\mathfrak{B}(t,x;M)\asymp|x|^{-\frac{\sigma}{1-m}}$. This proves that, for any choice of the parameters A,B,t_0 and for any choice of the mass M the inequality $V(t,x)>\mathfrak{B}(t,x;M)$ holds $|x|$ large enough.
As a final remark, we can define another family of subsolutions. Indeed, for some choice of the parameters B',F' and T the function W defined as

$$W(t,x)=\frac{(T-t)^{\frac{1}{1-m}}}{(B'+F'|x|^\sigma)^\alpha}\,,$$

is a subsolution to the equation (CP) which has the same qualitative behaviour as $|x| \to \infty$ but also the drawback of the extinction in finite time.

Proof of Proposition 4.4.1: We just need to verify that the function $\underline{V}(t,x)$ defined in (4.4.2) satisfies the inequality $\partial_t \underline{V}(t,x) \le |x|^\gamma \mathrm{div}\left(|x|^{-\beta}\nabla \underline{V}^m\right)$. Assume that $r=|x|$, as an abuse of notation we will write $\underline{V}(t,x)=\underline{V}(t,r)$ and $\underline{V}(t,r)$ is understood as a radial function. We recall that the operator $\mathcal{L}_{\gamma,\beta}=|x|^\gamma\nabla\cdot\left(|x|^{-\beta}\nabla f\right)$ acts on a radial function $f(r)$ in the following way

$$\mathcal{L}_{\gamma,\beta}(f)=r^{\gamma-\beta}\left(f''(r)+\frac{(d-1-\beta)}{r}\,f'(r)\right)\,. \tag{4.4.4}$$

A simple computation shows the following identities

$$\begin{aligned}
\partial_t\underline{V}(t,r) &= \frac{-A\,\alpha\,\partial_t D(t)}{(D(t)+Fr^\sigma)^{\alpha+1}}\,,\\
\mathcal{L}_{\gamma,\beta}\left(\underline{V}(t,r)^m\right) &= \frac{-(\sigma\,\alpha m\,A^m\,F)}{(D(t)+Fr^\sigma)^{\alpha m+2}}\left[(d-\gamma)D(t)+F\,r^\sigma\left(-\sigma\,\alpha\,m+d-2-\beta\right)\right]\,.
\end{aligned}$$

The inequality $\partial_t V(t,r)\le\mathcal{L}_{\gamma,\beta}\left(V(t,r)^m\right)$ is satisfied if and only if

$$\begin{aligned}
\frac{-A\,\alpha\,\partial_t D(t)}{(D(t)+Fr^\sigma)^{\alpha+1}} &\le \frac{-(\sigma\,\alpha m\,A^m\,F)}{(D(t)+Fr^\sigma)^{\alpha m+2}}\left[(d-\gamma)D(t)+F\,r^\sigma\left(-\sigma\,\alpha\,m+d-2-\beta\right)\right]\iff\\
\partial_t D(t) &\ge \frac{\sigma\,m\,F\,A^{m-1}}{(D(t)+Fr^\sigma)^{\alpha(m-1)+1}}\left[(d-\gamma)D(t)+F\,r^\sigma\left(-\sigma\,\alpha\,m+d-2-\beta\right)\right]\,,
\end{aligned}\tag{4.4.5}$$

where to obtain the last line we have used the fact that $\alpha>0$. The reader may notice that if $\varepsilon<2/(1-m)-\mathfrak{n}$ then in the right-hand-side of inequality (4.4.5) the term $F\,r^\sigma\left(-\sigma\,\alpha\,m+d-2-\beta\right)$ is negative. A simple computation then shows that the supremum of the right-hand-side of inequality (4.4.5) is achieved at $r=0$. Hence inequality $\partial_t V(t,r)\le\mathcal{L}_{\gamma,\beta}\left(V(t,r)^m\right)$ will follow asking that

$$\begin{aligned}
\partial_t D(t) &\ge \sigma\,m\,F\,A^{m-1}\,(d-\gamma)\,D(t)^{\alpha\,(1-m)}\\
&=\sup_{r\ge0}\frac{\sigma\,m\,F\,A^{m-1}}{(D(t)+Fr^\sigma)^{\alpha(m-1)+1}}\left[(d-\gamma)D(t)+F\,r^\sigma\left(-\sigma\,\alpha\,m+d-2-\beta\right)\right]\,.
\end{aligned}$$

We conclude the proof observing that, for any $t_0\in\mathbb{R}^d$, such an inequality is satisfied by the function $D(t)$ defined in (4.4.1). $\square$

A revision of the proof of Proposition 4.4.1 reveals that the condition on ε for $\underline{V}$ to be a subsolution is actually

$$m\,\varepsilon\le\frac{2}{1-m}-\mathfrak{n}\,,$$

while for $\varepsilon>\frac{1}{m}\left(\frac{2}{1-m}-\mathfrak{n}\right)$, $\underline{V}$ ceases to be a subsolution. This has some remarkable consequences. The above threshold allows one to chose $\varepsilon>\frac{2}{1-m}-\mathfrak{n}$ and so to have subsolutions which do not belong to $\mathrm{L}^1_\gamma(\mathbb{R}^d)$. This is a remarkable fact since it shows that, in general, initial data $u_0\notin\mathrm{L}^1_\gamma(\mathbb{R}^d)$ produce solutions that will not be in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ for any time $t>0$. We resume this fact in the following corollary.

Corollary 4.4.3. *Let $m \in (\frac{\mathfrak{n}-2}{\mathfrak{n}}, 1)$, $\varepsilon \in \left[\frac{2}{1-m} - \mathfrak{n}, \frac{1}{m}(\frac{2}{1-m} - \mathfrak{n})\right]$, $A, B, t_0 > 0$ and $\alpha = \frac{1}{1-m} - \frac{\varepsilon}{2} > 0$. Then the function $V(t,x)$ defined in (4.4.2) satisfies inequality (4.4.3) and for any $t \geq 0$*

$$\underline{V}(t,\cdot) \notin \mathrm{L}^1_\gamma(\mathbb{R}^d)\,.$$

Moreover, if $u(t,x)$ is a solution to (CP) with initial data u_0 such that, for some $A, B, t_0 > 0$ and for some $\varepsilon \in \left[\frac{2}{1-m} - \mathfrak{n}, \frac{1}{m}(\frac{2}{1-m} - \mathfrak{n})\right]$ it holds

$$u_0(x) \geq \frac{A}{(D(t_0) + B|x|^\sigma)^\alpha}$$

then for any $t > 0$

$$u(t,\cdot) \notin \mathrm{L}^1_\gamma(\mathbb{R}^d)\,.$$

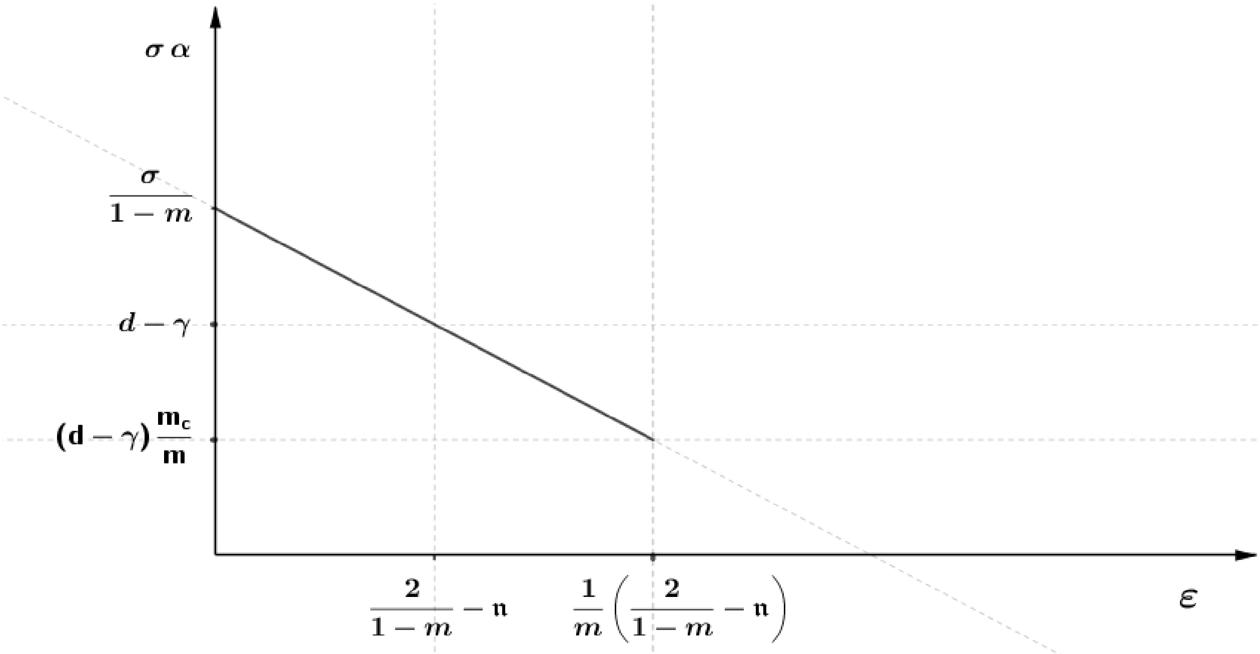

Figure 4.1: Tail behaviour of the subsolution $\underline{V}$ with respect to ε.

The family of subsolutions constructed in Proposition 4.4.1 can be classified according to its power-like behaviour at infinity. For $\varepsilon = 0$ the subsolution has a power-like tail which resembles the Barenblatt's one and goes to zero as $|x|^{\frac{-\sigma}{1-m}}$. As ε grows this behaviour changes, first $\sigma\alpha = \frac{\sigma}{1-m} - \frac{\sigma\varepsilon}{2}$ reaches the value of $d-\gamma$ when $\varepsilon = \frac{2}{1-m} - \mathfrak{n}$ and finally for $\varepsilon = \frac{1}{m}\left(\frac{2}{1-m} - \mathfrak{n}\right)$ we have that $\sigma\alpha = (d-\gamma)\frac{m_c}{m}$, see figure 4.1. We recall that we are not allowed to chose $\varepsilon > \frac{1}{m}\left(\frac{2}{1-m} - \mathfrak{n}\right)$, since the function $\underline{V}(t,x)$ defined in (4.4.2) ceases to be a subsolution for high value of ε. An intriguing fact is that we are indeed allowed to take negative values for ε, however for such a choice the subsolution will stay below a Barenblatt profile and will not provide any new information since inequality (4.2.2) holds for any solution which has initial data in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$. It is interesting to inquire the integrability of these subsolutions. As already noticed in Corollary (4.4.3) we have that for

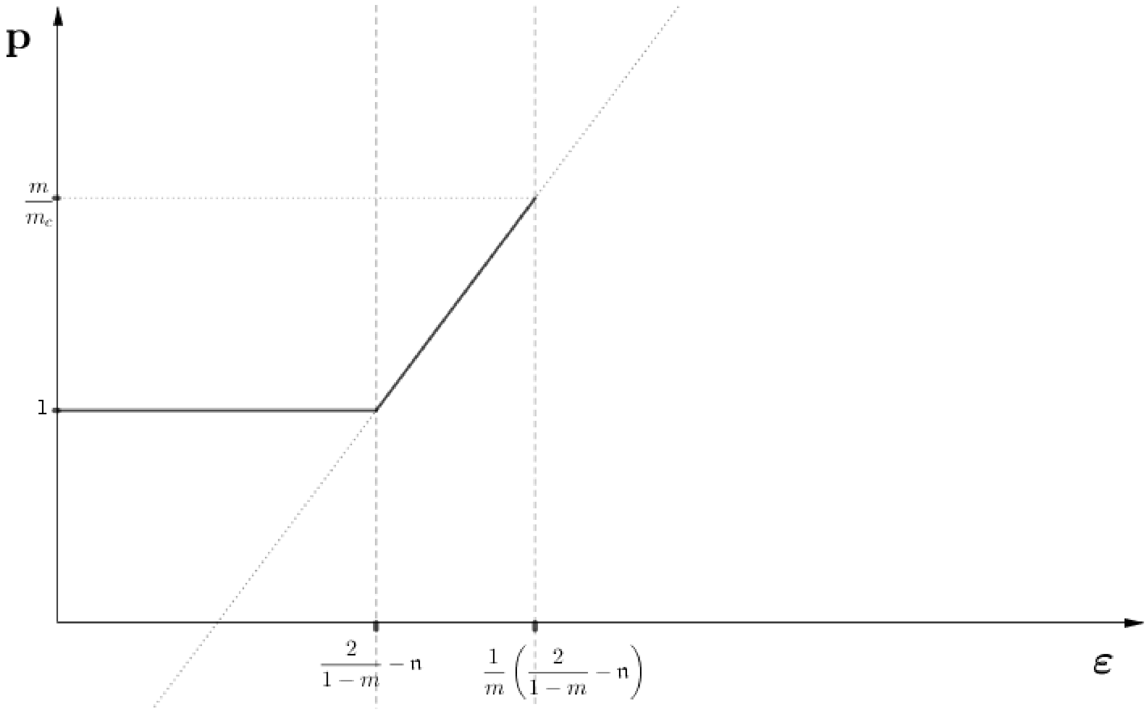

Figure 4.2: Integrability of the subsolution $\underline{V}$ with respect to ε.

any choice of $\varepsilon \in \left[0, \frac{2}{1-m} - \mathfrak{n}\right)$ the subsolution $\underline{V}(t,x)$ is in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$. The family does not belong to $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ for $\varepsilon > \frac{2}{1-m} - \mathfrak{n}$. If $\varepsilon = \frac{1}{m}\left(\frac{2}{1-m} - \mathfrak{n}\right)$ then for any $\delta > 0$ the subsolution $\underline{V}(t,x) \in \mathrm{L}^{\frac{m}{m_c}+d}_\gamma(\mathbb{R}^d)$ but $\underline{V}(t,x) \notin \mathrm{L}^{\frac{m}{m_c}}_\gamma(\mathbb{R}^d)$. See figure 4.2.

As a corollary of Proposition 4.4.1 we have the following.

Corollary 4.4.4. *Let $m \in (\frac{\mathfrak{n}-2}{\mathfrak{n}}, 1)$, $\varepsilon \in (0, \frac{2}{1-m} - \mathfrak{n})$, $A, B, C > 0$ and $\alpha = \frac{1}{1-m} - \frac{\varepsilon}{2} > 0$ and assume that*

$$u_0(x) \geq \frac{A}{(C + B|x|^\sigma)^\alpha}\,.$$

Then we have

$$\liminf_{|x|\to\infty} |x|^{\sigma\,\alpha}\, u(t,x) \geq \frac{A}{B}\,.$$

The proof of the above Corollary is an immediate application of Proposition 4.4.1. We conclude the section proving that if the initial datum u_0 does not satisfy the tail condition (TC) there is no chance to conclude that uniform convergence to the Barenblatt profile in *relative error* holds.

Corollary 4.4.5. *Under the same assumptions of Corollary 4.4.4 we have that for any $M > 0$ and for any $t > 0$*

$$\sup_{x\in\mathbb{R}^d} \left| \frac{u(t,x)}{\mathfrak{B}(t,x;M)} - 1 \right| = \infty\,. \tag{4.4.6}$$

Proof. Since

$$u_0(x) \geq \frac{A}{(C + B|x|^\sigma)^\alpha}\,,$$

we conclude by Proposition 4.4.1 that for any $t > 0$

$$u(t,x) \geq \frac{A}{(D(t) + B|x|^\sigma)^\alpha} ,$$

where $D(t) = \left(\sigma A^{m-1} m B (d-\gamma)(1-\alpha(1-m))\, t + C^{1-\alpha(1-m)}\right)^{\frac{1}{1-\alpha(1-m)}}$. A simple computation shows that the quotient

$$\frac{1}{\mathfrak{B}(t,x;M)} \frac{A}{(D(t) + B|x|^\sigma)^\alpha} = \frac{A}{t^{\frac{1}{1-m}}} \frac{\left[b_0 \frac{t^{\sigma\vartheta}}{M^{\sigma\vartheta(1-m)}} + b_1|x|^\sigma\right]^{\frac{1}{1-m}}}{(D(t) + B|x|^\sigma)^{\frac{1}{1-m}-\frac{\varepsilon}{1}}} \sim \frac{A\, b_1^{\frac{1}{1-m}}}{B^{\frac{1}{1-m}-\frac{\varepsilon}{2}}} \frac{|x|^{\frac{\sigma\varepsilon}{2}}}{t^{\frac{1}{1-m}}} \quad \text{as} \quad |x| \to \infty ,$$

from which we deduce (4.4.6). The proof is then complete. $\square$

4.4.2 Construction of a supersolution

In this section we construct a family of supersolutions which share same qualitative behaviour with the subsolutions constructed in Propositon 4.4.1.

Proposition 4.4.6. *Let $m \in (\frac{\mathfrak{n}-2}{\mathfrak{n}}, 1)$, $\varepsilon \in (0, \frac{2}{1-m} - \mathfrak{n})$, $E, F > 0$ and $\alpha = \frac{1}{1-m} - \frac{\varepsilon}{2} > 0$. Define for some $t_0 \in \mathbb{R}$ and $H > 0$ the function*

$$G(t) := t_0 + H\, t, \tag{4.4.7}$$

and let us define the function

$$\overline{V}(t,x) = \frac{E\, G(t)^\alpha}{(G(t) + F|x|^\sigma)^\alpha} \in \mathrm{L}^1_\gamma(\mathbb{R}^d) . \tag{4.4.8}$$

If H satisfies the following condition

$$H \geq m\, \sigma\, F^2\, E^{m-1}\, (2 + \beta - d + \sigma\, \alpha\, m) , \tag{4.4.9}$$

then

$$\partial_t \overline{V}(t,x) \geq |x|^\gamma \mathrm{div}\left(|x|^{-\beta} \nabla \overline{V}^m\right)(t,x) . \tag{4.4.10}$$

Proof of Proposition 4.4.6: We just need to verify that the function $\overline{V}(t,x)$ defined in (4.4.8) satisfies inequality (4.4.10) under the assumption (4.4.9). We shall assume that $r = |x|$, as an abuse of notation we will write $\overline{V}(t,x) = \overline{V}(t,r)$ and $\overline{V}(t,r)$ is understood as a radial function. We recall that the operator $\mathcal{L}_{\gamma,\beta} = |x|^\gamma \nabla \cdot \left(|x|^{-\beta} \nabla f\right)$ acts on a radial function $f(r)$ as written in (4.4.4). We have the following identities

$$\begin{aligned}
\partial_t \overline{V}(t,r) &= \frac{\alpha\, E\, G(t)^{\alpha-1}\, H}{(G(t) + F\, r^\sigma)^{\alpha+1}}\, F\, r^\sigma , \\
\mathcal{L}_{\gamma,\beta}\left(\overline{V}^m(t,r)\right) &= \frac{\sigma\, \alpha\, m\, F\, E^m G(t)^{\alpha\, m}}{(G(t) + F\, r^\sigma)^{\alpha m+2}} \left[F\, r^\sigma (2 + \beta - d + \sigma\, \alpha\, m) - (d-\gamma) G(t)\right] .
\end{aligned}$$

It is straightforward to verify that (4.4.10) holds at $r = 0$ since for any $t > 0$ the derivative in time $\partial_t \overline{V}(t,0) = 0$ and $\mathcal{L}_{\gamma,\beta}\left(\overline{V}^m(t,0)\right)$ is negative. When $r > 0$ a simple computation shows that (4.4.10) is equivalent to the following inequality

$$H \geq \left(\frac{G(t)}{G(t) + F\, r^\sigma}\right)^{1-\alpha(1-m)} m\, \sigma\, F\, E^{m-1} \left[F\, (2 + \beta - d + \sigma\, \alpha\, m) - (d-\gamma) \frac{G(t)}{r^\sigma}\right] . \tag{4.4.11}$$

Since $\left(\frac{G(t)}{G(t)+F\,r^\sigma}\right)^{1-\alpha(1-m)} < 1$ and $(d-\gamma)\frac{G(t)}{r^\sigma} > 0$, it is straightforward to see that (4.4.9) implies (4.4.11). Hence $\overline{V}(t,x)$ is a supersolution and the Proposition is proved. $\square$

The proof of the above Proposition reveals that if we chose $\varepsilon < 0$ then $\overline{V}(t,x)$ ceases to be a supersolution. Indeed, if $\varepsilon < 0$ we have that $1-\alpha(1-m) < 0$ and so in (4.4.11) we would have that

$$\left(\frac{G(t)}{G(t)+F\,r^\sigma}\right)^{1-\alpha(1-m)} = \left(1+\frac{F\,r^\sigma}{G(t)}\right)^{\alpha(1-m)-1} \to \infty \quad \text{as} \quad r \to \infty\,,$$

and an inequality as (4.4.11) would be impossible.

The supersolutions $\overline{V}$ constructed above share some features with the subsolutions constructed in Proposition 4.4.1: the have the same qualitative power-like behaviour for $|x| \to \infty$ and share the same integrability properties as those explained in fig. 4.2. We would like to add a few remarks.

Initial data which have a power-like decay for $|x| \to \infty$ such as

$$(A+|x|)^{-\alpha} \le u_0(x) \le (B+|x|)^{-\alpha}$$

where $\mathfrak{n} < \alpha < \sigma/(1-m)$ will produce a solution $u(t,x)$ which would have the same power-like behaviour for $|x| \to \infty$ for all times, so no change in the qualitative behaviour at infinity occurs. This is a remarkable fact since it shows a clear difference between the "good" range $(m_c, 1)$ and the range $(0, m_c)$, in the former one it has been shown the existence of initial data which produce solutions with a power-like behaviour for $|x| \to \infty$ which changes with time, see [143].

As we previously have proven, a GHP for general data is not possible, however a generalized GHP, in which the solution is sandwiched between two different power-like tails, can be still an option, see Section 4.4.3.

4.4.3 Generalized global Harnack principle

In the previous section we have proven that the GHP as stated in Theorem 4.1.1 simply cannot hold if we do not ask the correct decay for the initial data. Nevertheless we are able to prove a *generalized* version of the GHP when the initial data decay more slowly than the Barenblatt profile.

Theorem 4.4.7. *Let* $m \in (\frac{\mathfrak{n}-2}{\mathfrak{n}}, 1)$*,* $\varepsilon \in (0, \frac{2}{1-m} - \mathfrak{n})$ *and* $\alpha = \frac{1}{1-m} - \frac{\varepsilon}{2} > 0$*. Assume that the initial data* u_0 *satisfies*

$$\frac{A}{\left(\underline{t}^{\frac{1}{1-\alpha(1-m)}} + B\,r^\sigma\right)} \le u_0(x) \le \frac{E\,\overline{t}^\sigma}{\left(\overline{t}+F\,r^\sigma\right)^\alpha}\,,$$

for some $A, B, E, F, \underline{t}, \overline{t} > 0$*. Then for any* $t > 0$ *we have that*

$$\underline{V}(t,x) := \frac{A}{(D(t)+B|x|^\sigma)^\alpha} \le u(t,x) \le \frac{E\,G(t)^\alpha}{(G(t)+F|x|^\sigma)^\alpha} =: \overline{V}(t,x)$$

where

$$D(t) := \left(\sigma\,A^{m-1}\,m\,B\,(d-\gamma)\,(1-\alpha(1-m))\,t + \underline{t}\right)^{\frac{1}{1-\alpha(1-m)}} \quad \text{and} \quad G(t) := t_0 + H\,t,$$

where $H \ge m\,\sigma\,F^2\,E^{m-1}\,(2+\beta-d+\sigma\,\alpha\,m)$*.*

Proof of Theorem 4.4.7. The proof is a simple combination of the results contained in Proposition 4.4.1 and Proposition 4.4.6. $\square$

4.4.4 Final remarks

As we already mentioned, the convergence of solutions to (CP) to the Barenblatt profile has been studied by many researchers and many results are available. Indeed, we have shown in Theorem 4.2.4 that for data in $\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$ convergence to the Barenblatt profile holds in uniform relative error in parabolic cylinders. At a first sight the existence of the subsolution $\underline{V}$ of Proposition 4.4.1 might be in contrast with the above mentioned convergence result. However this is not the case and the family of sub/supersolutions give us a new insight on the mechanism of convergence to the Barenblatt profile for solutions whose initial datum does not satisfy (TC). Indeed let us consider a solution u which is sandwiched between $\underline{V}$ and $\overline{V}$ and for which (TC) does not hold. Let us as notice first that, even if both $\underline{V}$ and $\overline{V}$ give us a quite precise description of the tail behaviour, they are actually quite imprecise in describing the behaviour of u in compact sets. Indeed, on those sets $\underline{V}$ approaches zero uniformly while $\overline{V}$ goes uniformly to a constant. This is quite a different behaviour from the one that can be expected from Theorem 4.2.4. We then must concluded that neither $\underline{V}$ nor $\overline{V}$ are significative in describing the asymptotic behaviour of u in expanding sets of the form $\{|x| \le Ct^\vartheta\}$ or more generally on compact sets. Nevertheless the Barenblatt profile fails to describe uniformly the asymptotic behaviour of u in sets of the form $\{|x| \ge Ct^\vartheta\}$. We therefore expect that the correct asymptotic behaviour for u should be given by a different function which resembles the behaviour of the Barenblatt profile in expanding sets $\{|x| \le Ct^\vartheta\}$ while it has a proper decay in sets of the form $\{|x| \ge Ct^\vartheta\}$.

4.5 On the fast diffusion flow in $\mathcal{X}$

In this section we finally analyse the properties of the space $\mathcal{X}$. As we have already seen, $\mathcal{X}$ plays a key role in the proof of the upper estimates of Theorem 4.3.1 and it is the space where the *uniform convergence* in relative error takes place, therefore we believe that it is of paramount importance to give some detailed information about it.

4.5.1 Examples of functions in $\mathcal{X}$ which do not satisfy the pointwise decay condition $|f| \lesssim |x|^{-\frac{\sigma}{1-m}}$

We have already claimed that the norm of $\mathcal{X}$ measures how fast a function decays at infinity. More precisely, if $f \in \mathcal{X}$ then it decays faster or as the Barenblatt profile $\mathfrak{B}_M$, however we notice that the topology in which this decay is measured substantially differs from the one of L^∞ or L^1. Indeed, let us consider the following examples. To fix ideas let us take $\sigma = 2$ and $\gamma = \beta = 0$. Let N be an integer such that $N \ge 2$ and define $x_N = (N, \underline{0})$, where $\underline{0} \in \mathbb{R}^{d-1}$ is the zero vector of $\mathbb{R}^{d-1}$. Define the function f to be

$$f(y) = \frac{1}{N^{\frac{2}{1-m}-1}} \quad \text{if} \quad y \in B_{\frac{1}{N^2}}(x_N)\,, \qquad f(y) = 0 \quad \text{otherwise}\,.$$

It is straightforward to verify that $B_{N^{-2}}(x_N) \cap B_{M^{-2}}(x_M) \neq \emptyset$ if and only if $N = M$ therefore the definition makes sense. Let us now prove that the function f is integrable, indeed

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \sum_{N=2}^{\infty} \int_{B_{\frac{1}{N^2}}(x_N)} f(x)\,\mathrm{d}x = \sum_{N=2}^{\infty} \omega_d \frac{N^{-2d}}{N^{\frac{2}{1-m}-1}}\,,$$

the right-hand-side of the above expression converges if $\frac{2}{1-m} - 1 + 2d > 1$, which holds true since $d \geq 3$. We also have that

$$\int_{\mathbb{R}^d} |x|^{\frac{2}{1-m}} f(x)\,\mathrm{d}x = \sum_{N=2}^{\infty} \int_{B_{N-2}(x_N)} |x|^{\frac{2}{1-m}} f(x)\,\mathrm{d}x \leq \sum_{N=2}^{\infty} \omega_d 2^{\frac{2}{1-m}} \frac{N^{\frac{2}{1-m}-2d}}{N^{\frac{2}{1-m}-1}}$$
$$\leq \omega_d 2^{\frac{2}{1-m}} \sum_{N=2}^{\infty} \frac{1}{N^{2d-1}}\,,$$

where we have used the fact that if $y \in B_{\frac{1}{N^2}}(x_N)$ then $|y| \leq 2N$. So, by the following computation we have that $f \in \mathcal{X}$:

$$R^{\frac{2}{1-m}-d} \int_{B^c_R(0)} f\,\mathrm{d}x \leq R^{\frac{2}{1-m}} \int_{B^c_R(0)} f\,\mathrm{d}x \leq \int_{B^c_R(0)} |x|^{\frac{2}{1-m}} f\,\mathrm{d}x < \infty\,.$$

However it straightforward to verify that for any $A > 0$ there exists $\overline{N} = \overline{N}(A)$ such that for any $M \geq \overline{N}$ we have that

$$f(x) \geq \frac{A}{|x|^{\frac{2}{1-m}}}\,, \quad \text{for any} \quad x \in B_{\frac{1}{M^2}}(x_M)\,.$$

We have proven that

$$\text{for any } R > 0 \qquad \sup_{|x|\geq R} \frac{f(x)}{|x|^{\frac{2}{1-m}}} = \infty\,.$$

Therefore $f(x)$ does not decay at infinity as a Barenblatt profile. Here we give another counter example. Let $\alpha, \beta > 0$ and define the function g to be

$$g(y) = \frac{1}{|y - x_N|^{-\beta}} \quad \text{if} \quad y \in B_{N^{-\alpha}}(x_N)\,, \qquad f(y) = 0 \quad \text{otherwise}\,.$$

It is easy to see that if

$$d > \beta > 0 \quad \text{and} \quad \alpha > \max\left\{\frac{1}{d-\beta}\,, \frac{2}{d}\left(\frac{2}{1-m} - d + 1\right)\right\},$$

then $g \in \mathcal{X}$, however $g \notin \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$.

Finally we give a radial example. Let again $\alpha, \beta > 0$ such that $0 < \beta < 1$ and $r_N = N$ for any positive integer $N \geq 2$, and define the set $A_N = \{r_N \leq |x| \leq r_N + N^{-\alpha}\}$ and the function $h(x) = h(|x|)$ to be

$$h(y) = h(r) = \frac{1}{|r - r_N|^\beta} \quad \text{if} \quad y \in A_N\,, \qquad h(y) = 0 \quad \text{otherwise}\,.$$

If $\alpha > d/(1-\beta)$ we have that $h \in \mathrm{L}^1(\mathbb{R}^d)$, indeed

$$\int_{\mathbb{R}^d} h(x)\,\mathrm{d}x = \omega_d \sum_{N\geq 2} \int_{A_N} \frac{r^{d-1}\,\mathrm{d}r}{|r - r_N|^\beta} \leq \omega_d \sum_{N\geq 2} (2N)^{d-1} \int_{r_N}^{r_N+N^{-\alpha}} \frac{\mathrm{d}r}{|r - r_N|^\beta}$$
$$= \omega_d \sum_{N\geq 2} (2N)^{d-1} \int_0^{N^{-\alpha}} s^{-\beta}\,\mathrm{d}s = \omega_d \sum_{N\geq 2} (2N)^{d-1} N^{-\alpha(1-\beta)}\,,$$

where we have used the fact that if $y \in A_N$ then $|y| \leq 2N$. The series converges since $\alpha > d/(1-\beta)$. With the same technique it is easy to show that $h \in \mathcal{X}$ if

$$\alpha > \frac{2}{(1-m)(1-\beta)}\,.$$

4.5.2 Different norms on $\mathcal{X}$

The norm $|\cdot|_{\mathcal{X}}$ has some unpleasant features, indeed equipping $\mathcal{X}$ with $|\cdot|_{\mathcal{X}}$ would lead to a definition of a non-complete space. To see this we consider, for any $0<\varepsilon<\frac{\sigma}{1-m}-(d-\gamma)$, the function

$$f(x)=|x|^{-\frac{\sigma}{1-m}}\left(1-\chi_{B_1(0)}\right)+|x|^{-(d-\gamma)-\varepsilon}\,\chi_{B_1(0)}\,,$$

where $\chi_{B_1(0)}=1$ on $B_1(0)$ and vanishes outside $B_1(0)$. The function f does not belong to $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ nevertheless $|f|_{\mathcal{X}}<\infty$, and it can be approximated in the topology induced by $|\cdot|_{\mathcal{X}}$. To see this let us define, for any $0<r<1$ the family $\{f_r(x)\}$:

$$f_r(x)=|x|^{-\frac{\sigma}{1-m}}\left(1-\chi_{B_1(0)}\right)+|x|^{-(d-\gamma)-\varepsilon}\,\chi_{B_1(0)\setminus B_r(0)}\,.$$

We have that $f_r\in\mathcal{X}$ for any $0<r\le 1$ and a simple (but lengthy) computation shows that $|f_r-f|_{\mathcal{X}}\to 0$ as $r\to 0$. To avoid this phenomena we introduce the following, more suitable, norm on $\mathcal{X}$

$$\|f\|_{\mathcal{X}}:=\sup_{R>0}\,(1\vee R)^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)}\int_{B_R^c(0)}|f(x)||x|^{-\gamma}\,\mathrm{d}x<\infty\,. \tag{4.5.1}$$

The main difference between $|\cdot|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ is that the latter takes into account the influence of the $\mathrm{L}^1_\gamma(\mathbb{R}^d)$-norm, as it is proven in the following Proposition.

Proposition 4.5.1. *Assume that $\gamma<d$ and $m\in(m_c,1)$. Then,*

i) For any $f\in\mathcal{X}$ we have that

$$\|f\|_{\mathcal{X}}=\max\{\|f\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)},|f|_{\mathcal{X}}\}\,; \tag{4.5.2}$$

ii) $\mathcal{X}$ equipped with the norm $\|\cdot\|_{\mathcal{X}}$, defined in (4.5.1)*, is a Banach space;*

Proof. We will first prove $i)$. We first prove that $\|f\|_{\mathcal{X}}\le\max\{\|f\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)},|f|_{\mathcal{X}}\}$. For any $0\le R\le 1$ we have that $(1\vee R)^{\sigma/(1-m)-(d-\gamma)}\int_{B_R^c(0)}f|x|^{-\gamma}\,\mathrm{d}x\le\|f\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}$ while for any $R\ge 1$ we have that $(1\vee R)^{\sigma/(1-m)-(d-\gamma)}\int_{B_R^c(0)}f|x|^{-\gamma}\,\mathrm{d}x\le|f|_{\mathcal{X}}$, so we have proved the wanted inequality. We shall now prove that $\|f\|_{\mathcal{X}}\ge\max\{\|f\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)},|f|_{\mathcal{X}}\}$: we have that

$$\|f\|_{\mathcal{X}}\ge\lim_{R\to 0}\,(1\vee R)^{\sigma/(1-m)-(d-\gamma)}\int_{B_R^c(0)}f|x|^{-\gamma}\,\mathrm{d}x=\|f\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}\,.$$

Finally notice that

$$\begin{aligned}\|f\|_{\mathcal{X}}&=\sup_{R\ge 0}\,(1\vee R)^{\sigma/(1-m)-(d-\gamma)}\int_{B_R^c(0)}f(x)|x|^{-\gamma}\,\mathrm{d}x\\&\ge\sup_{R\ge 0}R^{\sigma/(1-m)-(d-\gamma)}\int_{B_R^c(0)}f(x)|x|^{-\gamma}\,\mathrm{d}x=|f|_{\mathcal{X}}\,.\end{aligned}$$

We shall now prove $ii)$. Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{X}$. By inequality (4.5.2) $\{u_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$. Since $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ is complete there exists a function $\overline{u}\in\mathrm{L}^1_\gamma(\mathbb{R}^d)$ such that $u_k\to\overline{u}$ in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$. Up to passing to a subsequence we can suppose that u_k converges to $\overline{u}$ also $a.e.$ in $\mathbb{R}^d$. We will prove that $\overline{u}\in\mathcal{X}$ and that u_k converges to $\overline{u}$ in $\mathcal{X}$. Let $N\ge 0$ be the smallest integer such that for any $k, m \geq N$ we have that $\|u_k - u_m\|_{\mathcal{X}} \leq 1$, as a consequence we then have that $\sup_{k\geq N} \|u_k\|_{\mathcal{X}} \leq 1 + \|u_N\|$. Let $R \geq 1$ and let $M = M(R)$ be the smallest integer such that for any $k \geq M$ we have that $\int_{\mathbb{R}^d} |u_k - \overline{u}| \,\mathrm{d}x \leq R^{d-2/(1-m)}$, recall that $d - 2/(1-m) < 0$. As a consequence of this choices by the triangle inequality we have that for any $k \geq \max\{N, M\}$

$$\int_{B_R^c(0)} |\overline{u}| \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_k - \overline{u}| \,\mathrm{d}x + \int_{B_R^c(0)} |u_k| \,\mathrm{d}x \leq R^{d-\frac{2}{1-m}} \left(2 + \|u_N\|_{\mathcal{X}}\right)$$

Since the choice of R was arbitrary we have proved that $\|\overline{u}\| \leq 2 + \|u_k\|_{\mathcal{X}}$. It remains to prove that u_k converges to $\overline{u}$ in the topology induced by $\|\cdot\|_{\mathcal{X}}$. Since $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$, for any $\varepsilon > 0$ there exists an integer $K > 0$ such that for any $k, m \geq K$ we have that, for any $R > 0$,

$$R^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_R^c(0)} |u_k - u_m| \,\mathrm{d}x < \varepsilon \,.$$

Taking the limit in $k \to \infty$ and applying the Fatou lemma to u_k we deduce from the above inequality that

$$R^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_R^c(0)} |\overline{u} - u_m| \,\mathrm{d}x < \varepsilon \,,$$

uniformly in $R > 0$, which is enough to conclude. The proof is then concluded. □

Remark 4.5.2. By definition, $\mathcal{X}$ is a proper subspace of $\mathrm{L}^1_\gamma(\mathbb{R}^d)$, however the topology induced by $\|\cdot\|_{\mathcal{X}}$ is quite different from the one induced by the $\mathrm{L}^1_\gamma(\mathbb{R}^d)$-norm. For example, it is not difficult to see that compactly supported functions are not dense in $\mathcal{X}$. Indeed, if we try to approximate $\mathfrak{B}_M$ with a compactly supported function we will face the following problem: let ϕ be a smooth compactly supported function such that $\mathrm{supp}(\phi) \subset B_{R_0}(0)$ for some $R_0 > 0$, then for any $R > R_0$ we have that

$$\int_{B_R^c(0)} |\mathfrak{B}_M - \phi| \,\mathrm{d}x = \int_{B_R^c(0)} \mathfrak{B}_M \,\mathrm{d}x \,,$$

and therefore

$$\|\mathfrak{B}_M - \phi\|_{\mathcal{X}} \geq \sup_{R\geq R_0} R^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_R^c(0)} \mathfrak{B}_M \,\mathrm{d}x \geq \left(\frac{R_0^\sigma}{R_0^\sigma + C(M)}\right)^{\frac{1}{1-m}} H \,,$$

where H is a positive constant which depends on d, m, γ, β, while $C(M)$ also depends on the mass M. The reader should not be surprised by this fact, since the norm $\|\cdot\|_{\mathcal{X}}$ takes into account the behaviour of the tails, it is quite natural that compactly supported functions shall not be a dense set.

It is interesting to notice that we can explicitly compute the values of $|\mathfrak{B}_M|_{\mathcal{X}}$ and $|\mathfrak{B}_{M,T}(t,\cdot)|_{\mathcal{X}}$. In what follows we will make use of the following auxiliary function

$$\mathcal{B}_\infty(x) = |x|^{-\frac{\sigma}{1-m}} \,. \tag{4.5.3}$$

Proposition 4.5.3. *Let $d \geq 3$, $\gamma < d$, $\beta \in \mathbb{R}^d$ such that $\gamma - 2 < \beta \leq \frac{d-2}{d}\gamma$ and $m \in (\frac{\mathfrak{n}-2}{\mathfrak{n}}, 1)$. Then the following identities hold*

i) The function $\mathcal{B}_\infty \notin \mathcal{X}$ and $\|\mathfrak{B}_M\|_{\mathcal{X}} = \max\{M, (1-m)\,\vartheta\,\omega_d\}$;

ii)

$$|\mathcal{B}_\infty|_{\mathcal{X}} = (1-m)\,\vartheta\,\omega_d \quad \textit{and} \quad |\mathfrak{B}_M|_{\mathcal{X}} = \lim_{R\to\infty} \frac{R^{\frac{\sigma}{1-m}}}{R^{(d-\gamma)}} \int_{B_R^c(0)} \mathfrak{B}_M \frac{\mathrm{d}x}{|x|^\gamma} = (1-m)\,\vartheta\,\omega_d\,;$$

iii) For any $M>0$ and for any $T\in\mathbb{R}$ we have, for any $t>\min\{|T|,0\}$, that

$$|\mathfrak{B}_{M,T}(t,\cdot)|_{\mathcal{X}} = \left(\frac{R_\star(t+T)}{\zeta}\right)^{\frac{1}{(1-m)\vartheta}} (1-m)\,\vartheta\,\omega_d\,, \tag{4.5.4}$$

where $\omega_d = |\mathcal{S}^{d-1}|$ is the measure of the $(d-1)$-dimensional sphere in $\mathbb{R}^d$.

Proof. Point $i)$ is a consequence of $iii)$ of Proposition 4.5.1.

Let us analyse point $ii)$. To compute $|\mathcal{B}_\infty|_{\mathcal{X}}$ we consider the following computation, for any $R>0$ we have that

$$R^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_R^c(0)} \mathcal{B}_\infty |x|^{-\gamma}\,\mathrm{d}x = R^{\frac{\sigma}{1-m}-(d-\gamma)}\,(1-m)\vartheta\,\omega_d R^{(d-\gamma)-\frac{\sigma}{1-m}} = (1-m)\,\vartheta\,\omega_d,$$

taking the supremum in $R>0$ we get the result. To obtain $|\mathfrak{B}_M|_{\mathcal{X}} = (1-m)\vartheta$ we first notice that $\mathfrak{B}_M \le \mathfrak{B}_\infty$, therefore $|\mathfrak{B}_M|_{\mathcal{X}} \le |\mathfrak{B}_\infty|_{\mathcal{X}}$. It only remains to prove the converse inequality: for any $R>0$ we have that

$$\begin{aligned} R^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_R^c(0)} \mathfrak{B}_M \frac{\mathrm{d}x}{|x|^\gamma} &\ge \left(\frac{R^\sigma}{R^\sigma + C(M)}\right)^{\frac{1}{1-m}} R^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_R^c(0)} \mathcal{B}_\infty \frac{\mathrm{d}x}{|x|^\gamma} \\ &= \left(\frac{R^\sigma}{R^\sigma + C(M)}\right)^{\frac{1}{1-m}} (1-m)\vartheta\,\omega_d\,, \end{aligned}$$

taking the supremum in $R>0$ we obtain $|\mathfrak{B}_M|_{\mathcal{X}} \ge |\mathfrak{B}_\infty|_{\mathcal{X}}$. From the above inequality it is clear that the supremum of the $|\cdot|_{\mathcal{X}}$ is assumed at infinity. Therefore $i)$ is proved.

To prove $iii)$ we notice that it is enough to prove the first identity in (4.5.4), since the second can be obtained as a consequence of $iii)$ in Proposition 4.5.1. Let $\rho>0$, we observe that by a change of variables we have that

$$\rho^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_\rho^c(0)} \mathfrak{B}_{M,T}(t,x)\frac{\mathrm{d}x}{|x|^\gamma} = \left(\frac{R_\star(t+T)}{\zeta}\right)^{\frac{1}{(1-m)\vartheta}} r^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{B_r^c(0)} \mathfrak{B}_M(y)\frac{\mathrm{d}y}{|y|^\gamma}\,,$$

which proves the first identity in (4.5.4). The proof is then concluded. □

4.5.3 The fast diffusion flow as a curve in $\mathcal{X}$

In the same spirit of [144] we want to investigate solutions to (CP) as curve in $\mathcal{X}$. As we already explained in Section 4.1 the space $\mathrm{L}^1_{\gamma,+}(\mathbb{R}^d)$ can be split into two different components $\mathcal{X}$ and $\mathcal{X}^c$. We have already shown that the flow is stable in $\mathcal{X}$, namely if $0\le u_0\in\mathcal{X}$ then the solution $u(t)\in\mathcal{X}$ for all times $t>0$. On the other hand, if $u_0\notin\mathcal{X}$ the flow will never enter $\mathcal{X}$, namely $u(t)\notin\mathcal{X}$ for all $t>0$, as it is proven in the following Proposition.

Proposition 4.5.4. *Let u,v be solutions to* (CP) *with initial data u_0,v_0 respectively. Then* *i) If* $0 \le u_0 \in \mathcal{X}$ *then for all* $t > 0\,,\ u(t,\cdot) \in \mathcal{X}\,,$

$$|u(t,\cdot) - v(t,\cdot)|_{\mathcal{X}} \le C\,(1 + t^{\frac{1}{1-m}})\,|u_0 - v_0|_{\mathcal{X}}\,. \tag{4.5.5}$$

ii) If $u_0 \ge 0$ *and* $u_0 \notin \mathcal{X}$ *then for all* $t > 0\,,\ u(t,\cdot) \notin \mathcal{X}\,.$

iii) If $u_0 \in \mathcal{X}^+$ *then* $u \in C\left((0,\infty);\mathcal{X}\right)$*. Moreover, if* $u_0 \in \mathcal{X}^+$ *and compactly supported we have that* $u \in C\left([0,\infty)\,;\mathcal{X}\right)$*;*

Remark 4.5.5. Assertions i) and ii) guarantee that the condition $u_0 \in \mathcal{X}$ (or $u(t_0) \in \mathcal{X}$) is always preserved under the flow, and if the initial data is not in $\mathcal{X}$ the solution will never be. Assertion i) shows also the continuous dependence on the initial data. In $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ the same property holds (it goes under the name of L^1_γ*-contraction*, namely $\|u(t,\cdot) - v(t,\cdot)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)} \le \|u_0 - v_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}$, see [18, Proposition 8]). We can conclude that there exists a constant $\mathcal{C}_{d,m,\gamma,\beta}$ such that

$$\|u(t,\cdot) - v(t,\cdot)\|_{\mathcal{X}} \le \mathcal{C}_{d,m,\gamma,\beta}\,(1 + t^{\frac{1}{1-m}})\,\|u_0 - v_0\|_{\mathcal{X}}\,.$$

Assertion iii) shows that the solution $u(t,\cdot)$ is a continuous path in $\mathcal{X}$ for any $t > 0$. The natural claim would be that $u \in C([0,\infty)\,,\mathcal{X})$ for any initial data in $\mathcal{X}$. Unfortunately we were not able to prove it neither disprove it for general initial data, and proving the assertion for compactly supported data is not enough to conclude (see Remark 4.5.2). However, we believe that this property holds and the strategy to prove it would be to prove that $-|x|^\gamma\nabla(|x|^{-\beta}\,\nabla u^m)$ is an m-accretive operator in $\mathcal{X}$ and then to apply the semigroup theory as done in [145]. This is a tedious task which falls out from the goal of this thesis. A final comment on the hypothesis on the positivity of the initial data u_0: the results of Proposition 4.5.4 holds even if the initial data is sign changing, we simply do not consider such solutions here.

Proof. The proof is based on the following inequality: for any $R > 0$ and for any $t, s \ge 0$ we have that

$$\left(\int_{B^c_R(0)} u(t,x)\frac{\mathrm{d}x}{|x|^\gamma}\right)^{1-m} \le \left(\int_{B^c_{2R}(0)} u(s,x)\frac{\mathrm{d}x}{|x|^\gamma}\right)^{1-m} + C\,|t-s|\,R^{(d-\gamma)(1-m)-\sigma}\,, \tag{4.5.6}$$

where C is a positive constant which depends on d, m, γ, β. The proof of (4.5.6) follows the line of Lemma 1.2.1 of Part I, Chapter 1: we only need to change the test function from a smooth, compactly supported function ϕ to $\psi = 1 - \phi$ and integrate on the complement of $B_R(0)$ instead of taking the integrals in $B_R(0)$. Since the solutions are in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$ integrating on the complement of a ball does not represent a problem. By taking the sup in $R > 0$ in inequality (4.5.6) we can deduce that for any $t, s \ge 0$ we have that

$$|u(t,\cdot)|_{\mathcal{X}} \le \mathfrak{C}\left(|u(s,\cdot)|_{\mathcal{X}} + t^{\frac{1}{1-m}}\right)\,, \tag{4.5.7}$$

where $\mathfrak{C}$ is a positive constant which depends on d, m, γ, β. From (4.5.7) we deduce that if $u_0 \in \mathcal{X}$ then for all $t > 0\,,\quad |u(t,\cdot)|_{\mathcal{X}} \le \mathfrak{C}\left(|u_0|_{\mathcal{X}} + t^{\frac{1}{1-m}}\right)$, where we have used inequality (4.5.7) with $s = 0$. In a similar manner we can deduce ii) from (4.5.7) by interchanging the roles of t and s.

It remains to prove iii), we adapt here the strategy of [146, Theorem 4.4]. Let $\phi(x) = \phi(|x|) \in C^\infty(\mathbb{R}^d)$ such that $\phi(|x|) = 1$ for $|x| \ge 2$, $\phi(|x|) = 0$ for $|x| \le 1$ and $\phi(|x|) > 0$ for $1 < |x| < 2$. We will prove that the quantity

$$R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\Big|u(t,x) - u(s,x)\Big|\phi\left(\frac{x}{R}\right)|x|^{-\gamma}\,\mathrm{d}x$$

is continuous in t uniformly in $R>0$. We divide the proof in two steps: $0<R<1$ and $R\ge 1$. In the case $0<R<1$ the results follows by strong continuity in $\mathrm{L}^1_\gamma(\mathbb{R}^d)$. By construction a solution u to (CP) belongs to $C\left([0,\infty)\,;\mathrm{L}^1_\gamma(\mathbb{R}^d)\right)$, see [18, Section 2.2]. Therefore for any $0\le R\le 1$ we have that

$$R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\Big|u(t,x)-u(s,x)\Big|\phi\left(\frac{x}{R}\right)\frac{\mathrm{d}x}{|x|^\gamma}\le\|u(t,\cdot)-u(s,\cdot)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}\to 0\quad\text{as}\quad t\to s\,,$$

which shows the wanted result in this case. It remains to consider $R\ge 1$. The Aronson-Benilan estimate $(1-m)u_t\le u/t$ implies that the function $t^{-\frac{1}{1-m}}u(t,x)$ is nonincreasing in t for any $x\in\mathbb{R}^d$. Let $0<t_0\le t_1$, then we have that

$$u(t_0,x)-u(t_1,x)\ge\left[\left(\frac{t_0}{t_1}\right)^{\frac{1}{1-m}}-1\right]u(t_0,x)=-\left[1-\left(\frac{t_0}{t_1}\right)^{\frac{1}{1-m}}\right]u(t_0,x)\,.$$

Define the negative part of a function f as $(f)_-:=\max\{-f,0\}$, the above inequality shows the following bound $(u(t_0,x)-u(t_1,x))_-\le\left[1-\left(\frac{t_0}{t_1}\right)^{\frac{1}{1-m}}\right]u(t_0,x)$, multiplying by $\phi(x/R)$ and integrating on $\mathbb{R}^d$ we get

$$\begin{aligned}&\int_{\mathbb{R}^d}(u(t_0,x)-u(t_1,x))_-\,\phi\left(\frac{x}{R}\right)|x|^{-\gamma}\,\mathrm{d}x\\&\le\left[1-\left(\frac{t_0}{t_1}\right)^{\frac{1}{1-m}}\right]\int_{\mathbb{R}^d}u(t_0,x)\phi\left(\frac{x}{R}\right)|x|^{-\gamma}\,\mathrm{d}x\,.\end{aligned}\tag{4.5.8}$$

Let us define the positive part of a function f as $(f)_+:=\max\{f,0\}$, we notice that $f=(f)_+-(f)_-$. Applying the identity $|f|=f+2(f)_-$ we deduce from the above inequality that for any $R>0$ we have

$$\begin{aligned}&R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\Big|u(t_0,x)-u(t_1,x)\Big|\phi\left(\frac{x}{R}\right)|x|^{-\gamma}\,\mathrm{d}x=\\&=R^{\frac{\sigma}{1-m}-(d-\gamma)}\left(\int_{\mathbb{R}^d}(u(t_0,x)-u(t_1,x))\,\phi\left(\frac{x}{R}\right)|x|^{-\gamma}\,\mathrm{d}x\right.\\&\left.+2\int_{\mathbb{R}^d}(u(t_0,x)-u(t_1,x))_-\,\phi\left(\frac{x}{R}\right)|x|^{-\gamma}\,\mathrm{d}x\right)=(I)+(II)\,,\end{aligned}$$

Inequality (4.5.8) shows that the second term (II) vanishes as $t_1\to t_0$ (or viceversa), therefore it remains to prove that (I) is continuous in t. From inequality (4.5.6) we deduce that for any $R>0$ and for any $0<t_0\le t_1$ we have that

$$\begin{aligned}&R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_0,x)\frac{\mathrm{d}x}{|x|^\gamma}\\&\le\left[\left(R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_1,x)\frac{\mathrm{d}x}{|x|^\gamma}\right)^{1-m}+C_1|t_0-t_1|\right]^{\frac{1}{1-m}}\,,\end{aligned}$$

therefore (I) can be estimated as follows

$$
\begin{aligned}
(I) &\leq \left(R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_0,x)\frac{\mathrm{d}x}{|x|^\gamma}\right)\\
&\quad\times\left[\left(1+\frac{R^{d-\gamma}\,C_1\,|t_0-t_1|}{R^{\frac{\sigma}{1-m}}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_1,x)\frac{\mathrm{d}x}{|x|^\gamma}}\right)^{\frac{1}{1-m}}-1\right]\\
&\leq |||u(t_1,\cdot)|||_{\mathcal{X}}\left[\left(1+\frac{R^{d-\gamma}\,C_1\,|t_0-t_1|}{R^{\frac{\sigma}{1-m}}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_1,x)\frac{\mathrm{d}x}{|x|^\gamma}}\right)^{\frac{1}{1-m}}-1\right],
\end{aligned}
\tag{4.5.9}
$$

where

$$
|||u(t_1,\cdot)|||_{\mathcal{X}}:=\sup_{R\geq 0}R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_0,x)\frac{\mathrm{d}x}{|x|^\gamma}\,.
$$

To conclude it remains to estimate $R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_1,x)\frac{\mathrm{d}x}{|x|^\gamma}$ by below uniformly in $R\geq 1$. In order to do so we invoke the result of Theorem 4.2.1: there exists $\underline{M}$ such that for any $t\geq t_0/2$ and for any $x\in\mathbb{R}^d$ we have that $u(t,x)\geq\mathfrak{B}(t-\frac{t_0}{4},x;\underline{M})$; from this last inequality we deduce that for any $R\geq 1$

$$
\begin{aligned}
\frac{R^{\frac{\sigma}{1-m}}}{R^{(d-\gamma)}}\int_{\mathbb{R}^d}\phi\left(\frac{x}{R}\right)\,u(t_1,x)\frac{\mathrm{d}x}{|x|^\gamma} &\geq \frac{R^{\frac{\sigma}{1-m}}}{R^{(d-\gamma)}}\int_{B^c_{2R}(0)}\mathfrak{B}(t_1-\frac{t_0}{4},x;\underline{M})\frac{\mathrm{d}x}{|x|^\gamma}\\
&\geq\mathcal{C}\,\frac{(t_1-\frac{t_0}{4})}{\left[b_0\frac{(t_1-\frac{t_0}{4})^{\sigma\vartheta}}{\underline{M}^{\sigma\vartheta(1-m)}}+b_1\right]^{\frac{1}{1-m}}}=:g(t_1)\,,
\end{aligned}
$$

where $\mathcal{C}$ is a constant which depends on d,m,γ,β and $g(t_1)$ converges to a positive constant $\kappa=\kappa(t_0)\neq 0$ as t_1 approaches t_0. Summarizing, we have that for any $R\geq 1$

$$
\begin{aligned}
&R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\Big|u(t_0,x)-u(t_1,x)\Big|\phi\left(\frac{x}{R}\right)\frac{\mathrm{d}x}{|x|^\gamma}\\
&\leq |||u(t_1)|||_{\mathcal{X}}\left[\left(1+\frac{C_1|t_0-t_1|}{g(t_1)}\right)^{\frac{1}{1-m}}-\left(\frac{t_0}{t_1}\right)^{\frac{1}{1-m}}\right],
\end{aligned}
$$

and the right-hand-side of the above inequality vanishes as $t_1\to t_0$. The first part of assertion $ii)$ is proved, it remains to prove continuity at $t=0$ for compactly supported data. Assume that u_0 is supported in $B_{R_0}(0)$ for some positive R_0. Then for any $R\leq 4\,R_0$ and for any $t\geq 0$ we have that

$$
\begin{aligned}
&R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\Big|u(t,x)-u_0(x)\Big|\phi\left(\frac{x}{R}\right)\frac{\mathrm{d}x}{|x|^\gamma}\\
&\leq(4R_0)^{\frac{\sigma}{1-m}-(d-\gamma)}\|u(t,\cdot)-u_0\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}\to 0\quad\text{as}\quad t\to 0\,.
\end{aligned}
$$

Let us consider now the case $R\geq 4\,R_0$, using inequality (4.5.6) and the fact that u_0 is supported in $B_{R_0}(0)$ we deduce

$$
\begin{aligned}
R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}\Big|u(t,x)-u_0(x)\Big|\phi\left(\frac{x}{R}\right)\frac{\mathrm{d}x}{|x|^\gamma} &= R^{\frac{\sigma}{1-m}-(d-\gamma)}\int_{\mathbb{R}^d}u(t,x)\phi\left(\frac{x}{R}\right)\frac{\mathrm{d}x}{|x|^\gamma}\\
&\leq 2^{\frac{1}{1-m}}\,C_1\,t^{\frac{1}{1-m}}\,,
\end{aligned}
$$

and the right-hand-side of the above inequality vanishes as t approaches 0. The proof is then concluded. □

The GHP (inequality (4.1.7)) allows one to give precise estimates on the behaviour of the norms $|\cdot|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ under the flow given by (CP). In the spirit of inequality (4.2.4) obtained in Corollary 4.2.2 we can compute the limit for $t \to \infty$ of both $|u(t,\cdot)|_{\mathcal{X}}$ and $\|u(t,\cdot)\|_{\mathcal{X}}$ when $u(t,x)$ is a solution to (CP) with initial data $u_0 \in \mathcal{X}^+$.

Proposition 4.5.6. *Under the assumption of Theorem 4.1.1 we have the following*

i) There exist $\overline{\tau}, \underline{\tau} > 0$ such that for t large enough we have that

$$\left(\frac{R_\star(t-\underline{\tau})}{\zeta}\right)^{\frac{1}{(1-m)\vartheta}} \le \frac{|u(t,\cdot)|_{\mathcal{X}}}{\omega_d\,(1-m)\,\vartheta} \le \left(\frac{R_\star(t+\overline{\tau})}{\zeta}\right)^{\frac{1}{(1-m)\vartheta}} . \tag{4.5.10}$$

ii) The following limits hold

$$\lim_{t\to\infty}\frac{|u(t,\cdot)|_{\mathcal{X}}}{t^{\frac{1}{1-m}}} = \lim_{t\to\infty}\frac{\|u(t,\cdot)\|_{\mathcal{X}}}{t^{\frac{1}{1-m}}} = (\sigma\, m)^{\frac{1}{1-m}}\left(\frac{\vartheta}{1-m}\right)^{\frac{m}{1-m}}\omega_d\,. \tag{4.5.11}$$

iii) For any $t>0$ the function $t \to \frac{|u(t,\cdot)|_{\mathcal{X}}}{t^{\frac{1}{1-m}}}$ is non increasing and for any $t \ge 0$ we have that

$$|u(t,\cdot)|_{\mathcal{X}} \ge \,(\sigma\, m)^{\frac{1}{1-m}}\left(\frac{\vartheta}{1-m}\right)^{\frac{m}{1-m}}\omega_d\, t^{\frac{1}{1-m}} \tag{4.5.12}$$

where $\omega_d = |\mathcal{S}^{d-1}|$ is the measure of the $(d-1)$-dimensional sphere in $\mathbb{R}^d$.

Remark 4.5.7. Inequality (4.5.12) obtained in *iii*) is sharp, in the sense that for any mass $M>0$ the Barenblatt profile $\mathfrak{B}(t,x;M)$ achieves the equality in (4.5.12).

Proof. *i*). Under the assumption of Theorem 4.1.1 we know that there exist $\underline{\tau}, \overline{\tau} > 0$ and $\underline{M}, \overline{M} > 0$ such that for any t large enough we have that

$$\mathfrak{B}_{\underline{M},-\underline{\tau}}(t,x) \le u(t,x) \le \mathfrak{B}_{\overline{M},\overline{\tau}}\,.$$

From the above inequality and inequality (4.5.4) of Proposition 4.5.3 one can easily deduce both inequalities in formula (4.5.10). Here we only remark that, since $R_\star(t)$ is increasing in t, for t large enough we have $\|\mathfrak{B}_M\| = \left(\frac{R_\star(t+T)}{\zeta}\right)^{\frac{1}{(1-m)\vartheta}}(1-m)\vartheta\,\omega_d$.

Identity (4.5.11) of *ii*) can be deduced from (4.5.10) once one sees that for any $T \in \mathbb{R}$ we have the following limit.

$$\lim_{t\to\infty} R_\star(t+T)^{\frac{1}{(1-m)\vartheta}}\, t^{-\frac{1}{1-m}} = \vartheta^{-\frac{1}{1-m}}\,.$$

The proof of the above limit is obtained by the definition of $R_\star$ in (4.1.3).

In order to prove *iii*) we introduce an auxiliary quantity which is easily differentiable under the flow. Let $k>0$ be a positive integer and $\phi_k(x)$ be such that

$$\phi_k(x) = 1 \text{ on } |x| \ge 1+\frac{1}{k}\,,\ \phi_k(x) = 0 \text{ on } |x| \le 1\,,\quad \text{and } \phi_k(x) > 0 \text{ on } 1 < |x| < 1+\frac{1}{k}\,.$$

Let us define

$$|||f|||_{k,\mathcal{X}} = \sup_{R>0} R^{\frac{\sigma}{1-m}-(d-\gamma)} \int_{\mathbb{R}^d} f(x)\,\phi_k\left(\frac{x}{R}\right)|x|^{-\gamma}\,\mathrm{d}x\,. \tag{4.5.13}$$

For any $k \geq 1$ and for any $f \in \mathcal{X}$ we have that

$$\left(\frac{k}{k+1}\right)^{\frac{\sigma}{1-m}-(d-\gamma)} |f|_{\mathcal{X}} \leq |||f|||_{k,\mathcal{X}} \leq |f|_{\mathcal{X}}\,.$$

As a consequence of the above inequality for any $f \in \mathcal{X}$ the following limit holds

$$\lim_{k\to\infty} |||f|||_{k,\mathcal{X}} = |f|_{\mathcal{X}}\,. \tag{4.5.14}$$

We take advantage of the auxiliary norms (4.5.13). Let $k > 0$ be a positive integer and $R > 0$ and define $Y_k(t) = \int_{\mathbb{R}^d} \phi_k\left(\frac{x}{R}\right) u(t,x)\frac{\mathrm{d}x}{|x|^\gamma}$. By the Aronson-Benilan estimate $u_t \leq \frac{u}{(1-m)t}$ we find that

$$Y_k'(t) \leq \frac{1}{(1-m)t}\,Y_k(t)\,,$$

which, integrated between any $\tau > s > 0$ leads to

$$\frac{Y_k(s)}{s^{\frac{1}{1-m}}} \geq \frac{Y_k(\tau)}{\tau^{\frac{1}{1-m}}}\,.$$

Multiplying by $R^{\sigma/(1-m)-(d-\gamma)}$ and taking the supremum in $R > 0$ in the above inequality we get

$$\frac{|||u(s,\cdot)|||_{k,\mathcal{X}}}{s^{\frac{1}{1-m}}} \geq \frac{|||u(t,\cdot)|||_{k,\mathcal{X}}}{t^{\frac{1}{1-m}}}\,,$$

taking the limit as $k \to \infty$ in the above inequality one gets the monotonicity of $t^{-1/(1-m)}|u(t,\cdot)|_{\mathcal{X}}$. Let $\tau > s > 0$ as before, from the inequality

$$\frac{|u(s,\cdot)|}{s^{\frac{1}{1-m}}} \geq \frac{|u(\tau,\cdot)|}{\tau^{\frac{1}{1-m}}}\,,$$

taking the limit as $\tau \to \infty$ and using (4.5.11) one obtains inequality (4.5.12). □

4.5.4 Convergence to the Barenblatt in $\mathcal{X}$

Finally we address here the question of convergence to the Barenblatt profile of solutions to (CP) in $\mathcal{X}$ with the topology induced by $\|\cdot\|_{\mathcal{X}}$. We find that it is false in general that

$$\|u(t,\cdot) - \mathfrak{B}(t,x;M)\|_{\mathcal{X}} \to 0 \quad \text{as} \quad t \to \infty\,.$$

To see this fact we provide an explicit counterexample. Consider the Barenblatt profile $\mathfrak{B}(t,x;M)$ and a translation in time of it $\mathfrak{B}(t+\tau,x;M)$. For $R > 0$ large enough we have that

$$\left|\mathfrak{B}(t+\tau,x;M) - \mathfrak{B}(t,x;M)\right| \geq \frac{1}{2}\left[\left(1+\frac{\tau}{t}\right)^{\frac{1}{1-m}} - 1\right]\mathfrak{B}(t,x;M) \quad \text{for any} \quad |x| \geq R\,,$$

we therefore conclude, thanks to identity (4.5.4), that

$$\left|\mathfrak{B}(t+\tau,x;M) - \mathfrak{B}(t,x;M)\right|_{\mathcal{X}} \gtrsim t^{\frac{m}{1-m}}\,.$$

However, if we suitably rescale the $\|\cdot\|_{\mathcal{X}}$ by $t^{\frac{1}{1-m}}$ we find the following result.

Proposition 4.5.8. *Under the assumption of Theorem 4.1.3 we have that*

$$\lim_{t\to\infty} \frac{\|u(t,\cdot)-\mathfrak{B}(t;M)\|_{\mathcal{X}}}{t^{\frac{1}{1-m}}} \to 0 \quad \textit{as} \quad t\to\infty\,. \tag{4.5.15}$$

Remark 4.5.9. It is interesting to notice that in *self-similar* variables, recall (4.1.15), the above convergence result takes place without the scaling factor $t^{\frac{1}{1-m}}$, namely we have that

$$\lim_{\tau\to\infty} \|v(\tau,\cdot)-\mathcal{B}_M(\tau)\|_{\mathcal{X}} = 0\,.$$

Proof. By Proposition 4.5.1 we know that $\|\cdot\|_{\mathcal{X}} = \max\{|\cdot|_{\mathcal{X}},\, \|\cdot\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}\}$, so to prove (4.5.8) we need to prove that both the limits

$$\lim_{t\to\infty} \frac{|u(t,\cdot)-\mathfrak{B}(t,\cdot;M)|_{\mathcal{X}}}{t^{\frac{1}{1-m}}} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\|u(t,\cdot)-\mathfrak{B}(t,\cdot;M)\|_{\mathrm{L}^1_\gamma(\mathbb{R}^d)}}{t^{\frac{1}{1-m}}} = 0\,,$$

hold. We will prove only the first one in the above formula since, due to the conservation of mass, the second one is trivial. Under the running assumption we know that $u(t,x)$ converge to the Barenblatt profile $\mathfrak{B}(t,x;M)$ in relative error. We restate this result in the following way: there exists a positive function $g(t):(0,\infty)\to(0,\infty)$ which $g(t)\to 0$ as $t\to\infty$ and such that for any $x\in\mathbb{R}^d$ and for any t large enough we have that

$$|u(t,x)-\mathfrak{B}(t,x;M)| \le g(t)\,\mathfrak{B}(t,x;M)\,.$$

By the above inequality we see that

$$\limsup_{t\to\infty} \frac{|u(t,\cdot)-\mathfrak{B}(t,\cdot;M)|_{\mathcal{X}}}{t^{\frac{1}{1-m}}} \le \lim_{t\to\infty} g(t)\,\frac{|\mathfrak{B}(t,\cdot;M)|_{\mathcal{X}}}{t^{\frac{1}{1-m}}} = 0\,,$$

where we have used identity (4.5.11) and the fact that $g(t)\to 0$ as $t\to\infty$. The proof is therefore concluded. $\square$

Appendix

4.6 Some results from Part I

For the sake of completeness we include here some results proven in Part I of this work. The following Theorem can be found in Chapter 1, Theorem 1.0.1 (inequality (1.0.1)).

Theorem 4.6.1 (Local Upper Bounds). *Let u be a nonnegative local strong solution to WFDE on the cylinder $\Omega \times (0,T]$, with initial data $u_0 \in \mathrm{L}^1_\gamma(\Omega)$. Let $m \in (m_c, 1)$ and $B_{2R_0}(x_0) \subset \Omega$ such that $R_0 = |x_0|/16$. Then there exist $\overline{\kappa}_1, \overline{\kappa}_2 > 0$ such that for any $t \in (0,T]$ we have*

$$\sup_{y \in B_{R_0}(x_0)} u\,(t,y) \le \frac{\overline{\kappa}_1}{t^{(d-\gamma)\vartheta}} \left[\int_{B_{2R_0}(x_0)} |u_0(y)|\ |y|^{-\gamma}\ \mathrm{d}y\right]^{\sigma\vartheta} + \overline{\kappa}_2 \left[\frac{t}{R_0^\sigma}\right]^{\frac{1}{1-m}}, \tag{4.6.1}$$

where ϑ and σ are defined in (4.1.2). The constants $\overline{\kappa}_1, \overline{\kappa}_2$ depend only on d, γ and β.

A simple corollary of the previous Theorem is the so-called smoothing effect for the Cauchy problem.

Theorem 4.6.2 (Upper Bounds for The Cauchy Problem). *Let $m \in (m_c, 1)$ and u be a nonnegative local strong solution to CP with $u_0 \in \mathrm{L}^1_\gamma(\mathbb{R}^d)$. Then there exist $\overline{\kappa}_1 > 0$ such that for any $t \in (0,\infty)$ we have*

$$\|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{\overline{\kappa}_1}{t^{(d-\gamma)\vartheta}} \left[\int_{\mathbb{R}^d} |u_0(y)|\ |y|^{-\gamma}\ \mathrm{d}y\right]^{\sigma\vartheta}, \tag{4.6.2}$$

where ϑ and σ are defined in (4.1.2). The constant $\overline{\kappa}_1$ depends only on d, γ and β.

In the proof of Theorem 4.2.1 in Section 4.2 we have made extensive use of the following result.

Theorem 4.6.3 (Local Lower Bounds). *Let $m \in (m_c, 1)$ and let u be a solution to CP with initial data $u_0 \in \mathrm{L}^1_{\gamma,\mathrm{loc}}$ and let $R > 0$. Define t_* as*

$$t_* = t_*(u_0, R) = \kappa_* \|u_0\|^{1-m}_{\mathrm{L}^1_\gamma(B_R(0))} R^{\frac{1}{\vartheta}}\,. \tag{4.6.3}$$

Then, there exists $\underline{\kappa} = \underline{\kappa}\big(d, m, \gamma, \beta\big) > 0$ such that

$$\inf_{x \in B_{2R}(0)} u(t,x) \ge \underline{\kappa} \left[\frac{t}{R^\sigma}\right]^{\frac{1}{1-m}} \qquad \text{for any } t \in [0, t_*].$$

The constant κ_* depends on d, m, γ, β.

The above Theorem has been proven in Part I, Chapter 2, Theorem 2.0.1. Theorem 4.6.3 applied to a solution $u(t,x)$ to the (CP) on a ball $B_R(0)$ at time t_* reads as

$$\inf_{x \in B_{2R}(x_0)} u(t_*, x) \ge \underline{\kappa}_1 \frac{\|u_0\|_{\mathrm{L}^1_\gamma(B_R(0))}}{R^{d-\gamma}}, \tag{4.6.4}$$

where $\underline{\kappa}_1$ depends only on d, m, γ, β.

4.7 Interpolation inequalities on bounded domains

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $u : \Omega \to \mathbb{R}$ be a function and define for any $\nu \in (0,1)$

$$\lfloor u \rfloor_{C^\nu(\Omega)} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\nu} . \tag{4.7.1}$$

We say that $u \in C^\nu(\Omega)$ whenever $\lfloor u \rfloor_{C^\nu(\Omega)} < \infty$. Notice that $\lfloor u \rfloor_{C^\nu(\Omega)} = 0$ if and only if u is constant, since in what follows we need to use strictly positive quantities we shall use the following inequality which holds for $u \in C^\nu(\Omega)$

$$|u(x) - u(y)| \leq \left(1 + \lfloor u \rfloor_{C^\nu(\mathbb{R}^d)}\right) |x-y|^\nu . \tag{4.7.2}$$

Let $\Omega' \subset \Omega$ be a subdomain, we define the distance between Ω and Ω' as

$$\mathrm{dist}(\Omega, \Omega') = \inf_{\substack{x \in \partial\Omega, \\ y \in \partial\Omega'}} |x-y| ,$$

where $\partial\Omega$ is the boundary of Ω and $\partial\Omega'$ is the boundary of Ω'. The purpose of this section of the appendix is to prove the following lemma.

Lemma 4.7.1. *Let $p \geq 1$, $\nu \in (0,1)$ and $u : \Omega^d \to \mathbb{R}$ be a function such that $u \in \mathrm{L}^p_\gamma(\Omega) \cap C^\nu(\Omega)$.*

i) Assume $\gamma \leq 0$ and let $\Omega' \subset \Omega$ such that $\mathrm{dist}(\Omega, \Omega') > 0$, then there exists a positive constant $C_{d,\gamma,\nu,p}$, which depends on d, γ, p and ν, such that

$$\|u\|_{\mathrm{L}^\infty(\Omega')} \leq C_{d,\gamma,\nu,p} \left(1 + \frac{\|u\|_{\mathrm{L}^p_\gamma(\Omega)}}{\left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right) \mathrm{dist}(\Omega, \Omega')^{\frac{1}{p}}}\right)^{\frac{d-\gamma}{d-\gamma+p\nu}} \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^{\frac{d-\gamma}{d-\gamma+p\nu}} \|u\|_{\mathrm{L}^p_\gamma(\Omega)}^{\frac{p\nu}{d-\gamma+p\nu}} . \tag{4.7.3}$$

ii) Assume $\gamma > 0$, $\mathrm{dist}(\Omega, \Omega') > 1$ and $\|u\|^p_{\mathrm{L}^p_\gamma(\Omega)} \leq \frac{p\nu}{d}$, then there exists a positive constant $C_{d,\gamma,\nu,p}$, which depends on d, γ, p and ν, such that

$$\|u\|_{\mathrm{L}^\infty(\Omega')} \leq C_{d,\gamma,\nu,p} \left(1 + \sup_{x \in \Omega'} |x|\right)^{\frac{\gamma}{p}} \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^{\frac{d}{d+p\nu}} \|u\|_{\mathrm{L}^p_\gamma(\Omega)}^{\frac{p\nu}{d+p\nu}} , \tag{4.7.4}$$

Proof. For any $x, y \in \Omega'$ we have, by the triangle inequality, that

$$|u(x)|^p \leq (|u(x) - u(y)| + |u(y)|)^p \leq 2^p \left(|u(x) - u(y)|^p + |u(y)|^p\right) .$$

Let $0 \leq R < \mathrm{dist}(\Omega, \Omega')$, averaging on a ball $B_R(x)$ we have

$$\begin{aligned} |u(x)|^p &\leq \frac{2^p}{\mu_\gamma(B_R(x))} \int_{B_R(x)} |u(x) - u(y)|^p |y|^{-\gamma} \, \mathrm{d}y + \frac{2^p}{\mu_\gamma(B_R(x))} \int_{B_R(x)} |u(y)|^p |y|^{-\gamma} \, \mathrm{d}y \\ &\leq 2^p R^{p\nu} \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^p + 2^p \frac{\|u\|^p_{\mathrm{L}^p_\gamma(\Omega)}}{\mu_\gamma(B_R(x))} , \end{aligned} \tag{4.7.5}$$

where in the last step we have used (4.7.2) and that $\int_{B_R(x)} |u(y)|^p |y|^{-\gamma}\,\mathrm{d}y \le \|u\|^p_{\mathrm{L}^p_\gamma(\Omega)}$. We claim that for any $x_0 \in \mathbb{R}^d$ and for any $R \ge 0$ there exist positive constants $c_{\gamma,d}, C_{\gamma,d}$ such that

$$c_{\gamma,d}\, R^d \left(|x_0| \vee \frac{R}{2}\right)^{-\gamma} \le \mu_\gamma(B_R(x)) \le C_{\gamma,d}\, R^d \left(|x_0| \vee \frac{R}{2}\right)^{-\gamma}. \tag{4.7.6}$$

The above inequality can be proven using the techniques developed in [139, Lemma 5.2, Appendix B], we will not include the proof here.

Let us now prove *i*). Assume that $\gamma \le 0$, from (4.7.6) we deduce that $\mu_\gamma(B_R(x)) \ge c_{\gamma,d}\, R^{d-\gamma}$, so from (4.7.5) we obtain

$$|u(x)|^p \le C \left(R^{p\nu}\, \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^p + R^{\gamma-d}\, \|u\|^p_{\mathrm{L}^p_\gamma(\Omega)}\right), \tag{4.7.7}$$

the right hand side of the above inequality achieves its minimum at

$$R_\star = \left(\frac{(d-\gamma)\, \|u\|^p_{\mathrm{L}^p_\gamma(\Omega)}}{p\nu\, \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^p}\right)^{\frac{1}{d-\gamma+p\nu}},$$

but a priori it can be that $R_\star \ge \mathrm{dist}(\Omega, \Omega')$ so we are not allowed to evaluate the right-hand side of (4.7.7) at $R_\star$. We instead evaluate it at $R = \delta\, R_\star$ where $\delta = \left(1 \wedge \frac{R_\star}{\mathrm{dist}(\Omega,\Omega')}\right) \le 1$. By simple computations we then find (4.7.3).

It only remains to prove *ii*). Assume that $0 < \gamma < d$, then using (4.7.6) in (4.7.5) we have for any $0 \le R < 1 < \mathrm{dist}(\Omega, \Omega')$

$$\begin{aligned}
|u(x)|^p &\le C \left(R^{p\nu}\, \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^p + \frac{\left(|x_0| \vee \frac{R}{2}\right)^\gamma}{R^d}\, \|u\|^p_{\mathrm{L}^p_\gamma(\Omega)}\right) \\
&\le C\,(1 + |x_0|^\gamma) \left(R^{p\nu}\, \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^p + R^{-d}\, \|u\|^p_{\mathrm{L}^p_\gamma(\Omega)}\right)
\end{aligned}$$

where in the last step we have used that $\left(|x_0| \vee \frac{R}{2}\right)^\gamma \le (1 + |x_0|^\gamma)$. Let

$$R_{\star\star} = \left(\frac{d\, \|u\|^p_{\mathrm{L}^p_\gamma(\Omega)}}{p\nu\, \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^p}\right)^{\frac{1}{d+p\nu}},$$

notice that by the assumption $\|u\|^p_{\mathrm{L}^p_\gamma(\Omega)} \le \frac{p\nu}{d}$ we have that $R_{\star\star} < 1$. After evaluating (4.7.8) at $R_{\star\star}$ and performing some simple computations we have that

$$|u(x_0)| \le C_{d,\gamma,\nu,p}\, (1 + |x_0|)^{\frac{\gamma}{p}}\, \left(1 + \lfloor u \rfloor_{C^\nu(\Omega)}\right)^{\frac{d}{d+p\nu}}\, \|u\|^{\frac{p\nu}{d+p\nu}}_{\mathrm{L}^p_\gamma(\Omega)},$$

taking the supremum in $x_0 \in \Omega'$ we get (4.7.4). The proof is then complete. □

4.8 Hölder continuity of solutions to weighted linear equations

We present here some regularity results for nonnegative local weak solutions to both linear and nolinear parabolic equations with weights. The results contained in this section are mainly contained in Part I, Chapter 3, Section 3.1 and references therein.

Consider the equation

$$v_t = w_\gamma \sum_{i,j=1}^{N} \partial_i \left(A_{i,j}(t,x)\, \partial_j v\right), \tag{4.8.1}$$

posed on the cylinder $Q := (0,T)\times\Omega$, where $A_{i,j} = A_{j,i}$ and there exist constants $0 < \lambda_0 \le \lambda_1 < +\infty$ such that for some $\gamma, \beta < N$, satisfying $\gamma - 2 < \beta \le \left(\frac{N-2}{N}\right)\gamma$, we have for any $\xi \in \mathbb{R}^d$ and any $x \in \mathbb{R}^d$

$$w_\gamma \asymp |x|^\gamma \qquad \text{and} \qquad 0 < \lambda_0 |x|^{-\beta}|\xi|^2 \le \sum_{i,j=1}^{N} A_{i,j}(t,x)\xi_i\xi_j \le \lambda_1 |x|^{-\beta}|\xi|^2 . \tag{4.8.2}$$

We shall consider bounded, nonnegative, local weak solutions to equation (4.8.1). For a precise definition we refer to [139, 65, 72] and references therein, here we just want to stress the fact that this class of solutions is large enough for our purposes.

It is convenient to introduce the notion of distance between sets of the form $Q = (0,T)\times\Omega \subset (0,\infty)\times\mathbb{R}^N$. Let $Q' = (T_1,T_2)\times\Omega' \subset Q$, we define

$$d_{\gamma,\beta}(Q,Q') := \inf_{\substack{(t,x)\in\{[0,T]\times\partial\Omega\}\cup\{\{0\}\times\Omega\},\\ (s,y)\in Q'}} |x-y| \vee \left(\rho_y^{\gamma,\beta}\right)^{-1}(|t-s|)\,, \tag{4.8.3}$$

where $\left(\rho_y^{\gamma,\beta}\right)^{-1}$ is the inverse of the following function $\rho_y^{\gamma,\beta}$ defined for any $y \in \mathbb{R}^d$ and γ, β as above

$$\rho_y^{\gamma,\beta}(R) := \left(\int_{B_R(y)} |x|^{(\beta-\gamma)\frac{N}{2}}\, \mathrm{d}x\right)^{\frac{2}{N}} .$$

Finally, we introduce the notion of C^α norm which takes into account the presence of the weights. With the above notation we define

$$\lfloor u \rfloor_{C^\alpha_{\gamma,\beta}(Q)} := \sup_{\substack{(t,x),(\tau,y)\in Q'\\ (t,x),\neq(\tau,y)}} \frac{|v(t,x)-v(\tau,y)|}{(|x-y| + |t-s|^{\frac{1}{2\vee\sigma}})^\alpha} . \tag{4.8.4}$$

The proof of the following result can be found in [139, Proposition 4.2, Corollary 4.3].

Proposition 4.8.1 (Hölder continuity for linear equations with weights)**.** *Let v be a nonnegative bounded local weak solution to equation (4.8.1) on $Q := (0,T)\times\Omega$, under the assumption (4.8.2). Let $Q' := (T_1,T_2)\times\Omega' \subset Q$. Then there exist $\alpha \in (0,1)$ and $\overline{\kappa}_\alpha > 0$, such that for all $(t,x),(s,y) \in Q'$*

$$\lfloor u \rfloor_{C^\alpha_{\gamma,\beta}(Q')} \le \frac{\overline{\kappa}_\alpha}{d_{\gamma,\beta}(Q,Q')^\alpha}\|v\|_{\mathrm{L}^\infty(Q)}, \tag{4.8.5}$$

where $\overline{\kappa}_\alpha > 0$ is given by

$$\overline{\kappa}_\alpha = \overline{\kappa}'_\alpha \begin{cases} 1\,, & \text{if } \sigma \ge 2,\\ \left(T^{\frac{1}{\sigma}} \vee \sup_{x_0\in\Omega}|x_0|\right)^{\frac{\gamma-\beta}{2}}, & \text{if } 0<\sigma<2\,. \end{cases} \tag{4.8.6}$$

The constants $\alpha, \overline{\kappa}'_\alpha$ depend only on $N, \gamma, \beta, \lambda_0, \lambda_1$.

Proposition 4.8.5 can be fruitfully used to deduce regularity results also for nonlinear parabolic equations: for example we can consider nonnegative bounded solutions to $u_t = |x|^\gamma \nabla\left(|x|^{-\beta}\nabla u^m\right)$ as solutions to the linear equation $u_t = |x|^\gamma \nabla\left(|x|^{-\beta}\, a(t,x)\,\nabla u\right)$ where $a(t,x) = mu(t,x)^{m-1}$. Indeed the same can be done for solutions to (4.1.16).

Lemma 4.8.2. *Let $\rho, \tau_0 > 0$, $0 < \lambda_0 \le \lambda_1 < \infty$, $m \in (0,1)$ and let $v(\tau, y) : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ be a nonnegative bounded solution to* (4.1.16) *and assume that*

$$\lambda_0 \le m\, v^{m-1}(\tau, y) \le \lambda_1 \quad \text{for any } \tau \ge \tau_0 \text{ and } \quad |y| \le \rho\,.$$

Then there exist $\nu > 0$ and $\overline{\kappa} > 0$ such that if $\tau_1 > \tau_0$ and $\tau \in \left[\tau_1 + \frac{1}{\sigma}\log R_\star(2), \tau_1 + \frac{1}{\sigma}\log R_\star(3)\right]$ then

$$\lfloor v(\tau,\cdot)\rfloor_{C^\nu\left(B_{\rho/2}(0)\right)} \le \overline{\kappa}\, \|v\|_{\mathrm{L}^\infty\left(\left[\tau_1, \tau_1 + \frac{1}{\sigma}\log R_\star(4)\right]\times B_\rho(0)\right)}\,. \tag{4.8.7}$$

The constants $\nu, \overline{\kappa}$ depend on $m, d, \gamma, \beta, \lambda_0, \lambda_1$; $\overline{\kappa}$ depends also on ρ.

Below we give the proofs of the results of this section.

Proof of Lemma 4.8.2. The proof is divided in several steps. We consider the function $\overline{v}(\tau, y)$ defined by

$$\overline{v}\,(\tau, y) := v\,(\tau + \tau_1) \quad \text{for any } \tau \ge 0\,.$$

Rescaling to original variables. The rescaled function $\overline{u}(t,x)$ defined by

$$\overline{u}(t,x) := \frac{\zeta^{d-\gamma}}{R_\star(t+1)^{d-\gamma}}\overline{v}\left(\frac{1}{\sigma}\log\frac{R_\star(t+1)}{R_\star(1)}, \frac{\zeta x}{R_\star(t)}\right) = \frac{\zeta^{d-\gamma}}{R_\star(t+1)^{d-\gamma}}\overline{v}\,(\tau, y) \tag{4.8.8}$$

satisfies (CP). Define the following domains

$$\overline{Q}_1 := \left\{(t,x) : 0 \le t \le 3, |x| \le \frac{\rho\, R_\star(t+1)}{\zeta}\right\}\,,\quad \overline{Q}_2 := \left\{(t,x) : 1 \le t \le 2, |x| \le \frac{\rho\, R_\star(t+1)}{2\,\zeta}\right\}$$

On both $\overline{Q}_1$ and $\overline{Q}_2$ the following estimate holds true

$$\frac{R_\star(1)^{(d-\gamma)(1-m)}}{\zeta^{(d-\gamma)(1-m)}}\,\lambda_0 \le m\tilde{u}^{m-1}(t,x) \le \lambda_1\,\frac{R_\star(4)^{(d-\gamma)(1-m)}}{\zeta^{(d-\gamma)(1-m)}}\,.$$

Application of the linear result. We can consider $\overline{u}$ as a bounded solution to the the linear equation

$$\overline{u}_t = |x|^\gamma\,\nabla\left(|x|^{-\beta}\, a(t,x)\,\nabla\overline{u}\right) \quad \text{where} \quad a(t,x) = m\overline{u}(t,x)\,,$$

on the domain $\overline{Q}_1$. From Proposition 4.8.1 we deduce that there exists $\nu > 0$ and $\overline{\kappa}_\nu > 0$ such that

$$\|\overline{u}\|_{C^\nu_{\gamma,\beta}(\overline{Q}_2)} \le \overline{\kappa}_\nu \frac{\|\overline{u}\|_{\mathrm{L}^\infty(\overline{Q}_1)}}{d_{\gamma,\beta}(\overline{Q}_1, \overline{Q}_2)^\nu}\,. \tag{4.8.9}$$

Since $R_\star(1), R_\star(4), \zeta$ are numerical constants which depend only on d, m, γ, β, the constant ν shall depend only on d, m, γ, β and λ_0, λ_1. However the constant $\overline{\kappa}_\nu$ will depend on ρ when $0 < \sigma < 2$, see the expression of the constant $\overline{\kappa}_\alpha$ in Proposition 4.8.1. Finally, we notice that $d_{\gamma,\beta}(\overline{Q}_1, \overline{Q}_2)^\nu$ depend as well on ρ. We shall now freeze the time variable and consider $\overline{u}(t,x)$ as a function in space only. From (4.8.9) we deduce that for any $t \in [1,2]$ we have that

$$\lfloor u(t,\cdot)\rfloor_{C^\nu\left(B_{\frac{\rho\,R_\star(t+1)}{2\,\zeta}}(0)\right)} \le \overline{\kappa}_\nu \frac{\|\overline{u}\|_{\mathrm{L}^\infty(\overline{Q}_1)}}{d_{\gamma,\beta}(\overline{Q}_1, \overline{Q}_2)^\nu}\,. \tag{4.8.10}$$

Rescaling back to self-similar variables. The domains $\overline{Q}_1$ ($\overline{Q}_2$ resp.) will be transformed back to $\big[\tau_1, \tau_1 + \frac{1}{\sigma}\log R_\star(4)\big] \times B_\rho(0)$ ($\big[\tau_1 + \frac{1}{\sigma}\log R_\star(2), \tau_1 + \frac{1}{\sigma}\log R_\star(3)\big] \times B_{\rho/2}(0)$ resp.). Estimate (4.8.10) becomes (4.8.7) where

$$\overline{\kappa} = \left(\frac{R_\star(4)}{R_\star(2)}\right)^{d-\gamma} \left(\frac{\zeta}{R_\star(1)}\right)^{\nu} \frac{\overline{\kappa}_\nu}{d_{\gamma,\beta}(\overline{Q}_1, \overline{Q}_2)^\nu}\,.$$

The proof is then concluded. □

Chapter 5

Global Harnack Principle for the p-Laplace evolution equation

In this chapter we investigate the global behaviour of solutions to the *fast p-Laplace* evolution equation $u_t = \Delta_p u$, where

$$\Delta_p u = \nabla.\left(|\nabla u|^{p-2}\nabla u\right) \,,$$

with $\frac{2d}{d+1} < p < 2$. We prove sharp, global lower bounds for solution to the Cauchy Problem with non-negative initial data $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. If the initial data also satisfies an additional condition ($u_0 \in \mathcal{X}_p$, where $\mathcal{X}_p$ is a suitable generalization of the same space introduce in Chapter 4): we prove that such lower estimates are matched by global upper estimates and the so called *Global Harnack Principle* holds. As a consequence, we prove that the convergence to the Barenblatt profile uniformly in relative error holds if and only if the initial data $u_0 \in \mathcal{X}$. Finally, exploiting a radial transformation introduced in [130] we prove sharp decay properties for the gradient of radial solutions in $\mathcal{X}$.

5.1 Introduction and statement of the main results

In this chapter we consider the initial value problem

$$\begin{cases} u_t(t,x) = \Delta_p u(t,x) & \text{for } t>0 \text{ and } x \in \mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{p-CP}$$

in the fast diffusion range $\frac{2N}{N+2} < p < 2$. The initial data is supposed to satisfy $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and the p-Laplace operator is defined as $\Delta_p w := \mathrm{div}(|\nabla w|^{p-2}\nabla w)$.

The theory of diffusion equations has been widely investigated in the last fifty years, however not a lot of the recent literature has been dedicated to the above Cauchy Problem. It is known that p-CP is solvable if $0 \le u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, weak solutions are unique and uniformly bounded if $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, [78] and [105]. It is known that the mass is conserved along the flow, however we will also give a proof of this fact. Existence of *self-similar* solutions is also known, they go under the name of *Barenblatt* solutions and assume the form $\mathcal{B}(t,x) = t^{-\alpha}F(|x|t^{-\beta})$ such that

$$\mathcal{B}(t,x;M) = t^{\frac{1}{2-p}}\left[b_1 \frac{t^{\frac{\beta p}{p-1}}}{M^{(2-p)\frac{\beta p}{p-1}}} + b_2\,|x|^{\frac{p}{p-1}}\right]^{-\frac{p-1}{2-p}}, \tag{5.1.1}$$

where b_1, b_2 depend on d, p; $\mathcal{B}(t, x; M)$ takes the Dirac delta with mass M as initial data:

$$\lim_{t\to 0} \mathcal{B}(t,x) = M\delta_0(x). \tag{5.1.2}$$

Our main goal is to prove quantitative lower bounds for the positive solutions of Problem (p-CP) and to introduce a suitable subspace of $\mathrm{L}^1(\mathbb{R}^d)$ where such lower bounds are matched by upper bound of the same type. As a consequence we characterize solutions which converge to the *Barenblatt* profile in *relative error.*

Notion of solution. We begin by giving the notion of weak solutions for the problem (p-CP).

Definition 5.1.1. *A solution to* (p-CP) *in* $(0,\infty)\times\mathbb{R}^d$ *is a measurable function*

$$u \in C_{\rm loc}\Big(\,[0,T]\,;\mathrm{L}^1_{\rm loc}(\mathbb{R}^d)\Big) \cap \mathrm{L}^p_{\rm loc}\left([0,T]; W^{1,p}_{\rm loc}(\mathbb{R}^d)\right)\,, \quad \textit{for any} \quad 0<T<\infty\,,$$

such that for any open $\Omega\subset\mathbb{R}^d$ *and for any interval* $[s,t]\subset[0,T]$ *the following equality holds true*

$$\int_{\mathbb{R}^d} u(s,x)\phi(s,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u(t,x)\phi(t,x)\,\mathrm{d}x + \int_s^t\int_{\mathbb{R}^d}\left(-u\phi_\tau + |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\right)\,\mathrm{d}x\,\mathrm{d}\tau\,, \tag{5.1.3}$$

for any test function $\phi \in W^{1,2}\Big([0,T];\mathrm{L}^2(\Omega)\Big)\cap \mathrm{L}^p(\mathbb{R}^d)\left([0,T];W^{1,p}_0(\Omega)\right)$.

The above definition is well known in the literature and can be found in [105] and [78]. We recall that, as proven in [78], weak solutions enjoy far more regularity than the one asked in the above definition. If the initial data $0\le u_0\in\mathrm{L}^1(\mathbb{R}^d)$ then solutions are locally $C^{1,\alpha}_{\rm loc}(\mathbb{R}^d)$ for some positive α which depends on d, p, see [78, Theorem III.8.1].

Let us introduce some parameters which will be widely used along this chapter:

$$\alpha = \frac{d}{d(p-2)+p}\,,\quad \beta=\frac{\alpha}{d}=\frac{1}{d(p-2)+p}\,,\quad p_c=\frac{2d}{d+1}\,. \tag{5.1.4}$$

The space $\mathcal{X}_p$. Before stating our result we need to introduce a suitable subspace of $\mathrm{L}^1(\mathbb{R}^d)$ for which our results hold. Let $f\in\mathrm{L}^1(\mathbb{R}^d)$ and $p\in(p_c,2)$. We say that $f\in\mathcal{X}_p$ if it satisfies

$$|f|_{\mathcal{X}_p} := \sup_{R>0} R^{\frac{1}{\beta(2-p)}}\int_{B^c_R(0)}|f(x)|\,\mathrm{d}x<\infty\,. \tag{5.1.5}$$

With a slight abuse of language we will also call $\mathcal{X}$ the tail space. Recall that for $p\in(p_c,2)$ we have that $\beta(2-p)>0$. It is easily seen that $|\cdot|_{\mathcal{X}_p}$ is a norm. Intuitively the quantity $|f|_{\mathcal{X}_p}$ measures *how fast* the function f decays at ∞ relatively to the decay of the Barenblatt profile $\mathcal{B}$. We now introduce a subspace of $\mathrm{L}^1(\mathbb{R}^d)$ of functions that satisfy the tail condition (5.1.5), that will play a key role in the rest of the paper:

$$\mathcal{X}_p := \{u\in\mathrm{L}^1(\mathbb{R}^d) : |u|_{\mathcal{X}_p}<+\infty\}.$$

Statement of the main results. Our first main result regards matching lower and upper bounds for solution to (p-CP). We have the following Theorem.

Theorem 5.1.2. *Let u be the solution of Problem* (p-CP) *and let* $0\le u_0\in\mathcal{X}_p\setminus\{0\}$. *Then for any* $t_0>0$ *there exist* $\underline{\tau},\overline{\tau}>0$ *and* $\underline{M},\overline{M}>0$ *such that*

$$\mathcal{B}(t-\underline{\tau},x;\underline{M})\le u(t,x)\le\mathcal{B}(t+\overline{\tau},x;\overline{M})\quad \textit{for all } x\in\mathbb{R}^d\,,\ t\ge t_0\,. \tag{5.1.6}$$

The values of $\underline{\tau},\underline{M}$ *and of* $\overline{\tau},\overline{M}$ *are given in the proof of Theorem 5.2.2, Theorem 5.3.1 respectively.*

Remark 5.1.3. The proof of the above result will be split into two cases: the upper bound, Theorem 5.3.1, and the lower bound, Theorem 5.2.2. For the upper bound the hypothesis $0 \le u_0 \in \mathcal{X}_p$, i.e. that u_0 has a precise decay at infinity, is strictly necessary. Indeed, for data $u_0 \notin \mathcal{X}_p$, as we already did in chapter 4, it is possible to construct explicit (sub)solutions that provide precise lower bounds that clearly contradict the upper bound of formula (5.1.6). More precisely, for any $t > 0$ and for any $x \in \mathbb{R}^d$ we have that

$$u(t,x) \ge \frac{1}{\left(D(t) + |x|^{\frac{p}{p-1}}\right)^{\frac{p-1}{2-p}-\varepsilon}} \gneqq \mathcal{B}(t,x;M)\,,$$

where $\varepsilon > 0$ is small, and $D(t) \sim t^{\frac{2}{\varepsilon(2-p)}}$. On the other hand, such hypothesis is not necessary for the lower bound of formula (5.1.6): indeed, lower bounds hold for any data $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, see Theorem 5.2.2.

It is known that solutions to (p-CP) with initial mass $M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}$ converge to the Barenblatt profile $\mathcal{B}(t,x;M)$ pointwise, uniformly in compact sets and in any L^q, for $1 \le q \le \infty$. On the other hand much less is known for *uniform* convergence in relative error. Theorem (5.1.2) allows us to characterize solutions that converge in relative error *uniformly* on $\mathbb{R}^d$. Indeed, we have the following Theorem.

Theorem 5.1.4. *Let $u(t,x)$ be the solution of Problem* (p-CP) *with initial data $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. Then the following limit holds*

$$\lim_{t\to\infty} \left\| \frac{u(t,x)}{\mathcal{B}(t,x;M)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \longrightarrow 0\,, \qquad \textit{where} \quad M = \|u_0\|_{\mathrm{L}}^1(\mathbb{R}^d)\,,$$

if and only if

$$u_0 \in \mathcal{X}_p \setminus \{0\}\,.$$

For the Fast Diffusion Equation ($u_t = \Delta u^m$) similar results can be found in [118, 106]. For problem (p-CP), to the best of our knowledge, the only similar result is contained in [147], and it is done for $p \ge 2$.

Remark 5.1.5. We will prove the above Theorem by means of other techniques, however, it is interesting to notice that it is possible to construct explicit solutions $u(t,x)$ for which

$$\sup_{x\in\mathbb{R}^d} \frac{u(t,x)}{\mathcal{B}(t,x;M)} = +\infty\,.$$

Organization of this chapter. Let us briefly explain in which order we will present the above results. In Section 5.2 we establish lower bounds for solutions to (p-CP). In Theorem 5.2.1 we prove local positivity estimates which will be suitably used in Theorem 5.2.2 to prove the lower bound of inequality (5.1.6). We conclude that section giving a proof of the conservation of mass. In Section 5.3 we prove first the upper bound of inequality (5.1.6) and finally we establish Theorem 5.1.4.

Some useful properties and known results. Solutions to (p-CP) posses a family of symmetries, in particular we will extensively make use the following rescaling property. If $v(t,x)$ is a solution to the (p-CP) with mass M, that is $\int_{\mathbb{R}^d} v_0(x)dx = M$ and

$$\begin{cases} v_t(t,x) = \Delta_p v(t,x) & \text{for } t > 0 \text{ and } x \in \mathbb{R}^d, \\ v(0,x) = v_0(x) & \text{for } x \in \mathbb{R}^d, \end{cases}$$

then

$$u(t,x) = \frac{1}{M} v(tM^{2-p}, x) \tag{5.1.7}$$

is a solution to the PLE with mass 1, that is $\int_{\mathbb{R}^d} u_0(x)dx = 1$ and

$$\begin{cases} u_t(t,x) = \Delta_p u(t,x) & \text{for } t > 0 \text{ and } x \in \mathbb{R}^d, \\ u(0,x) = \frac{1}{M} v_0(x) & \text{for } x \in \mathbb{R}^d, \end{cases}$$

Therefore, it is sufficient to work with solutions having mass 1 and then using relation (5.1.7) we recover the result for solutions with mass M.

In what follows we report some known and very useful a priori estimates for the problem (p-CP). Any solution u to (p-CP) satisfies the **Benilan-Crandall estimate**

$$u_t(t,x) \le \frac{u(t,x)}{(2-p)t} \,. \tag{5.1.8}$$

As consequence of (5.1.8) the following monotonicity estimates holds

$$t \to t^{-\frac{1}{2-p}} u(t,x) \quad \text{ is a decreasing function for any } x \in \mathbb{R}^d \,,\ t \in (0,\infty) \,. \tag{5.1.9}$$

As well, from (5.1.8) we can deduce that

$$\|u_t\|_{L^1(\mathbb{R}^d)} \le \frac{2(2-p)}{t} \|u_0\|_{L^1(\mathbb{R}^d)} \,. \tag{5.1.10}$$

Solutions to (p-CP) enjoys, sometime, more regularity than their initial data. The smoothing effect $\mathrm{L}^1 - -\mathrm{L}^\infty$ has been proved for instance in [48, Theorem 11.2], for the sake of clarity we report the statement of that Theorem below.

Theorem 5.1.6 (Smoothing effect). *Let $2 > p > \frac{2N}{N+1}$ and let u be the solution to equation* (p-CP) *with initial datum $0 \le u_0 \in L^1(\mathbb{R}^d)$. Then there is a constant $c(p,N)$ such that*

$$0 \le u(t,x) \le C_1 \|u_0\|^{p\beta}_{L^1(\mathbb{R}^d)} t^{-N\beta} \qquad \textit{for every } t > 0 \textit{ and for all } x \in \mathbb{R}^d \,, \tag{5.1.11}$$

where β is as in (5.1.4)*. The best constant $C_1 = C_1(p,N)$ is computed in [48, Theorem 11.2].*

In what follows we will make extensive use of the following **local-smoothing-effect**: let $x_0 \in \mathbb{R}^d$ and $R_0 > 0$, then any solution to (p-CP) satisfies the following inequality

$$\sup_{B_{R_0}(x_0)} u(t,x) \le \kappa_1 \frac{\left(\int_{B_{2R_0}(x_0)} u_0\right)^{p\beta}}{t^{N\beta}} + \kappa_2 \left(\frac{t}{R_0{}^p}\right)^{\frac{1}{2-p}} \qquad \text{for any } t > 0 \,. \tag{5.1.12}$$

The proof of the above inequality can be found in [75] or it [105, Thm. III.6.2].

The equation $u_t = \Delta_p u$ admits also a family of singular Barenblatt solutions $\mathcal{U}(x,t)$ which can be defined as the limit when $M \to \infty$ of the Barenblatt solution $\mathcal{B}(x,t;M)$. We write the singular Barenblatt solution starting at time S:

$$\mathcal{U}(x,t;S) = b_2^{-\frac{p-1}{2-p}} (t+S)^{\frac{1}{2-p}} |x|^{-\frac{p}{2-p}}. \tag{5.1.13}$$

We also will make use of the following modification of the Barenblatt solution:

$$\mathcal{U}(x,t;B_1,S) = (t+S)^{\frac{1}{2-p}} \left[b_2 \, |x|^{\frac{p}{p-1}} - B_1 (t+S)^{\frac{\beta p}{p-1}} \right]^{-\frac{p-1}{2-p}} . \tag{5.1.14}$$

This function has a singularity on the surface $|x| = R_U(t) = (B_1/b_2)^{(p-1)/p}(t+S)^\beta$ where the denominator vanishes and it is a solution of $u_t = \Delta_p u$ in the parabolic domain $\{|x| > R_U(t); t > 0\}$.

5.2 Lower bound for initial data in $\mathrm{L}^1_+(\mathbb{R}^d)$

In this Section we show a positivity estimate that holds for all data in $\mathrm{L}^1_+(\mathbb{R}^d)$, and as a consequence we deduce that the lower-part of the GHP estimates hold true (Theorem 5.2.2) for integrable data: this shows also a very interesting feature of the *fast diffusion regime*, i.e., the infinite speed of propagation, which can be also measured as "fatness of the tails".

5.2.1 Positivity estimates for initial data in $\mathrm{L}^1_+(\mathbb{R}^d)$

Here we consider solutions to (p-CP) and we obtain a local positivity estimate which holds for all times.

Theorem 5.2.1. *Let u be the solution of Problem* (p-CP) *and let $R>0$ and $x_0 \in \mathbb{R}^d$. Then*

$$\inf_{x\in B_{2R}(x_0)} u(t,x) \ge C\,\frac{M_R}{R^d}\begin{cases} \left(\frac{t}{t_c}\right)^{-N\beta} & \text{for } t\ge t_c\,, \\ \left(\frac{t}{t_c}\right)^{1/(2-p)} & \text{for } t\le t_c\,, \end{cases} \tag{5.2.1}$$

where

$$t_c = \kappa\, M_R^{2-p}\, R^{\frac{1}{\beta}} \quad \text{and} \quad M_R = \int_{B_R(x_0)} u_0 dx\,. \tag{5.2.2}$$

Proof. Without loss of generality we can assume that u_0 is supported in $B_R(x_0)$ and that $x_0=0$. If it is not the case, then define the function $v_0 = u_0\chi_{B_R(x_0)}$, where $\chi_{B_R(x_0)}=1$ on $B_R(x_0)$ and $\chi_{B_R(x_0)}=0$ outside $B_R(x_0)$, and let $v(t)$ be the solution to Problem (p-CP) with v_0 as its initial data. Then, by comparison, $u(t)\ge v(t)$ for any $t\ge0$, therefore applying the result to v will prove the result for u. As well, by scaling, we can assume that $\int_{B_R(x_0)} u_0 dx = 1$. We shall divide the proof in two steps.

Case $t<t_c$. We observe that for $t<t_c$ inequality (5.2.1) is a consequence of the Benilan-Crandall estimate (5.1.10) combined with (5.2.1) at $t=t_c$. Indeed, using the monotonicity (5.1.9) for $t\le t_c$, $u(t,x)$ satisfies

$$t^{-\frac{1}{2-p}}u(t,x) \ge t_c^{-\frac{1}{2-p}}u(t_c,x)\,.$$

By combining this with the lower bound at time t_c, in which case it just says that $u(x,t_c)\ge \frac{C}{R^d}$, we have that

$$u(t,x) \ge (t/t_c)^{\frac{1}{2-p}}u(t_c,x) \ge \frac{C}{R^d}(t/t_c)^{\frac{1}{2-p}}\,.$$

Case $t\ge t_c$ Let us define an auxiliary time $t_\star = \tilde\kappa\, R^{\frac{1}{\beta}}$ where $\tilde\kappa^{N\beta} = 2^{N+1}C_1\,|B_1|$ where C_1 is the constant of inequality (5.1.11) and $|B_1|$ is the volume of the ball of radius 1 in $\mathbb{R}^d$. We will first prove an initial lower bound for $u(t_\star,0)$ and then we shall generalize it to inequality (5.2.1). Let $r>0$ and $t\ge t_\star$, by conservation of mass we have that

$$1=\int_{\mathbb{R}^d} u(0,x)dx = \int_{\mathbb{R}^d} u(t,x)dx = \int_{B_{2R}} u(t,x)dx + \int_{B_{2R+r}\setminus B_{2R}} u(t,x)dx + \int_{\mathbb{R}^d\setminus B_{2R+r}} u(t,x)dx.$$

For the first integral we apply the smoothing effect (5.1.11):

$$\int_{B_{2R}} u(t,x)dx \le C_1 t^{-N\beta}|B_{2R}| = C_1 t^{-N\beta}|B_1|(2R)^N = C_1|B_1|\frac{(2R)^N}{t^{N\beta}}.$$

For the second integral we apply the Aleksandrov reflection principle ([110] Proposition A.1 page 425):

$$\int_{B_{2R+r}\setminus B_{2R}} u(t,x)dx \le |B_{2R+r}\setminus B_{2R}|\, u(0,t) = |B_1|((2R+r)^N - (2R)^N)u(0,t).$$

For the third integral we argue in the following way: for any $x_0 \in \mathbb{R}^d \setminus B_{2R+r}$ we use (5.1.12) on the ball $B_{R_0}(x_0)$ with $R_0 = |x_0|/2$. Since u_0 is supported in $B_R(x_0)$ the term $\int_{B_{R_0}(x_0)} u_0 dx = 0$ and so

$$u(t,x_0) \le \kappa_2\, 4^{\frac{p}{2-p}} \left(\frac{t}{|x_0|^p}\right)^{\frac{1}{2-p}}.$$

Applying the above inequality we get

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus B_{2R+r}} u(t,x)dx &\le \kappa_2\, 4^{\frac{p}{2-p}} \int_{\mathbb{R}^d\setminus B_{2R+r}} \left(\frac{t}{|x|^p}\right)^{\frac{1}{2-p}} \\
&\le \kappa_2\, 4^{\frac{p}{2-p}}\, t^{\frac{1}{2-p}}\, |\partial B_1| \int_{2R+r}^{\infty} r^{-\frac{p}{2-p}+N-1} dr \\
&\le \kappa_2\, 4^{\frac{p}{2-p}}\, t^{\frac{1}{2-p}}\, |\partial B_1|\, (2-p)\beta\, (2R+r)^{-\frac{1}{\beta(2-p)}}.
\end{aligned}$$

Combining all the previous estimates we arrive at

$$u(t,0) \ge \frac{f(r)}{|B_1|} := \frac{\left(B(t) - A(t)\,(2R+r)^{-\frac{1}{\beta(2-p)}}\right)}{((2R+r)^N - (2R)^N)}\, |B_1|^{-1} \tag{5.2.3}$$

where

$$B(t) = 1 - C_1\, |B_1|\, \frac{(2R)^N}{t^{N\beta}}, \quad \text{and} \quad A(t) = t^{\frac{1}{2-p}}\, \kappa_2\, 4^{\frac{p}{2-p}}\, |\partial B_1|\, (2-p)\beta.$$

We note that for $t \ge t_\star$ we have that $B(t) \ge 0$ and $B(t_\star) = 1/2$. The function $f(r)$ is continuous in r and the following equalities hold

$$\lim_{r\to 0^+} f(r) = -\infty \quad \text{and} \quad \lim_{r\to\infty} f(r) = 0.$$

Since $f(r)$ is sign changing we conclude that it has at least one local maximum, which we call r_m. At such point we have that $f'(r_m) = 0$, which translates into the following condition on r_m

$$\frac{A(t)}{\beta(2-p)} \left[(2R+r_m)^N - (2R)^N\right] = \left[B(t)\,(2R+r_m)^{\frac{1}{\beta(2-p)}} - A(t)\right] N\,(2R+r_m)^N. \tag{5.2.4}$$

By condition (5.2.4) we have that

$$f(r_m) = \frac{A(t)}{N\,\beta\,(2-p)\ (2R+r_m)^{\frac{p}{2-p}}}.$$

Again from (5.2.4) we deduce that

$$\frac{1}{(2R+r_m)^{\frac{p}{2-p}}} \ge \left(\frac{2-p}{p}\right)^{p\beta} \left(\frac{N\,B(t)}{A(t)}\right)^{p\beta},$$

and therefore

$$f(r_m) \ge \frac{(N(2-p))^{N\beta(2-p)}}{\beta\, p^{p\beta}}\, \frac{B(t)^{p\beta}}{A(t)^{N\beta(2-p)}}.$$

Finally we notice that $B(t) \geq B(t_\star)$ for any $t \geq t_\star$ and that $A(t) = \underline{c}t^{N\beta}$ which leads us to

$$u(t,0) \geq \frac{\underline{\kappa}}{R^N}\left(\frac{t_\star}{t}\right)^{N\beta}, \tag{5.2.5}$$

where

$$\underline{\kappa} = \frac{N^{N\beta(2-p)}}{\beta\,(2p)^{p\beta}\,\tilde{\kappa}^{N\beta}}\,\frac{|B_1|^{-1}}{\left(\kappa_2\,4^{\frac{p}{2-p}}\right)^{N\beta(2-p)}}\,.$$

Now we shall pass from the pointwise estimate at $x = 0$ to the infimum on the ball. The strategy will be to prove that for any $y \in \mathcal{B}_{2R}(0)$, $u(t,y)$ satisfies estimates (5.2.5) when $t \in t_\star$ and $t_\star$ does not depend on y. Let $y \in B_{2R}(0)$ and define $t_\star(y) = \tilde{\kappa}\,(4R)^{\frac{1}{\beta}}$, we apply the above mentioned procedure to the function u in the ball $B_{4R}(y)$ (notice than u_0 is supported in such a ball) and we get the following inequality

$$u(t,y) \geq \frac{\underline{\kappa}}{(4R)^N}\left(\frac{t_\star(y)}{t}\right)^{N\beta}, \quad \text{for any} \quad t \geq t_\star(y)\,. \tag{5.2.6}$$

We notice that for any $y, y_0 \in B_{2R}(0)$ the times $t_\star(y) = t_\star(y_0)$ (in other words for any $y \in B_{2R}(0)$ such a time is equal to a constant depending only on the radius of $B_{2R}(0)$) and therefore inequality (5.2.6) is uniform in $y \in B_{2R}(0)$. Taking the infimum (in $y \in B_{2R}(0)$) in (5.2.6) we get inequality (5.2.1) for any $t \geq t_c$ where t_c and the constant C of (5.2.1) have the following expression

$$t_c := 4^{\frac{1}{\beta}}\,\tilde{\kappa}\,R^{\frac{1}{\beta}} \quad \text{and} \quad \underline{C} = \underline{\kappa}\,4^{-N\left(\frac{\beta+1}{\beta}\right)}\,.$$

The proof is concluded once one rescales back to the original variables. □

5.2.2 A universal lower bound for initial data in $\mathrm{L}^1_+(\mathbb{R}^d)$

As a consequence of the local positivity result of Theorem (5.2.1) we are able to prove a *universal* (in the sense that it holds for any $\mathrm{L}^1(\mathbb{R}^d)$ data) and *global* lower bound (in the sense that it holds for all $x \in \mathbb{R}^d$).

Theorem 5.2.2. *Let u be the solution of Problem* (p-CP)*, let $t_0 > 0$ and $R > 0$ be such that $\|u_0\|_{\mathrm{L}^1(B_{R_0}(0))} > 0$. Then there exist $\underline{\tau} > 0$ and $\underline{M} > 0$ such that*

$$u(t,x) \geq \mathcal{B}(t-\underline{\tau}, x; \underline{M})\,, \quad \textit{for all } x \in \mathbb{R}^d \textit{ and any } t \geq t_0\,, \tag{5.2.7}$$

where

$$\underline{\tau} = \frac{1}{2}\,(t_c \wedge t_0) \quad \textit{and} \quad \underline{M} = b\,\|u_0\|_{\mathrm{L}^1(B_{R_0}(0))}\left(1 \wedge \frac{t_0}{t_c}\right)^{\frac{1}{1-m}} \tag{5.2.8}$$

where t_c is as in (5.2.2)*. The constant $b > 0$ depends only on d and p.*

Remark 5.2.3. The above Theorem reveals a remarkable property of solutions to (p-CP): the positivity spreads immediately for every nonnegative initial datum, showing infinite speed of propagation. A delicate issue is how to discriminate in a quantitative way among two infinite speeda of propagation. Our Theorem shows that we can put a (delayed) fundamental solution as a lower barrier for any data: this is how the PLE immediately creates a fat tail (inverse power), which is clearly bigger than the “standard Gaussian tail” created by the linear heat equation.

Proof of Theorem 5.2.2: Let us explain first the strategy of the proof. We will prove first that inequality (5.2.7) holds at time $t = t_c$ and then conclude discussing the two different cases, namely $t_0 \geq t_c$ and $0 < t_0 < t_c$.
Proof of inequality (5.2.7) *holds at time* $t = t_c$. More precisely we will prove that

$$u(t_c, x) \geq \mathcal{B}(t_c - \underline{\tau}, x; \underline{M}) , \quad \text{for all } x \in \mathbb{R}^d , \tag{5.2.9}$$

where

$$\underline{\tau} = a\, t_c \quad \text{and} \quad \underline{M} = b\, M_{R_0} ,$$

with $a \in (0,1)$ and $b > 0$ to be chosen later, t_c as in (5.2.2) and $M_{R_0} = \|u_0\|_{\mathrm{L}^1(B_{R_0}(0))}$. We split the proof in different steps. First, we find conditions (5.2.11) on a, b so that (5.2.9) holds in $|x| \leq R_0$. Next, we will find condition (5.2.14) on a, b so that (5.2.9) holds in $|x| \geq R_0$. Finally, we check the compatibility of all the conditions.
Condition inside a ball. We want to find condition on a, b such that the following inequality holds:

$$\underline{C}\,\frac{M_{R_0}}{{R_0}^d} \geq \frac{b^{p\,\beta} M_{R_0}}{b_1^{\frac{p-1}{2-p}} (1-a)^{d\beta} \kappa^{d\beta} {R_0}^d} = \sup_{x \in B_{2R_0}(0)} \mathcal{B}(t_c - \underline{\tau}, x; \underline{M}) , \tag{5.2.10}$$

where $\underline{C}$ is as in (5.2.1) and κ is as (5.2.2). It is easily seen that the former is implied by the following condition on a and b:

$$b^{p\,\beta} \leq \kappa^{d\,\beta}\, C\, b_1^{\frac{p-1}{2-p}} (1-a)^{d\,\beta} . \tag{5.2.11}$$

Note that by inequality (5.2.1) the first term in (5.2.10) is bounded above by $\inf_{x \in B_{2R_0}} u(t_c, x)$, therefore we obtain that

$$\inf_{x \in B_{2R_0}} u(t_c, x) \geq \sup_{x \in B_{2R_0}(0)} \mathcal{B}(t_c - \underline{\tau}, x; \underline{M}) ,$$

inequality (5.2.9) is then proved for any $|x| \leq 2R_0$.
Condition outside a ball. We want to find suitable conditions on a, b such that (5.2.9) holds in the outer region $|x| > R_0$. Such an inequality will be deduced by applying the comparison on the parabolic boundary of $Q = (\underline{\tau}, t_c) \times B^c_{R_0}(0)$, namely $\partial_p Q = \{\{\underline{\tau}\} \times B^c_{R_0}(0)\} \bigcup \{(\underline{\tau}, t_c) \times \{x \in \mathbb{R}^d : |x| = R_0\}\}$.
It is clear that $u(\underline{\tau}, x) \geq \mathcal{B}(0, x; \underline{M}) = \delta_0(x)$, for any $|x| \geq R_0$, hence we just need to prove that

$$u(t, x) \geq \mathcal{B}(t - \underline{\tau}, x; \underline{M}) \quad \text{for any} \quad |x| = R_0,\, t \in (\underline{\tau}, t_c) . \tag{5.2.12}$$

The following inequality

$$\underline{C}\, a^{\frac{1}{2-p}}\, \frac{M_{R_0}}{R_o^d} \geq \frac{b^{p\,\beta} M_{R_0}^{p\,\beta}}{b_1^{\frac{p-1}{2-p}} (1-a)^{d\,\beta}\, t_c^{d\,\beta}} , \tag{5.2.13}$$

implies that inequality (5.2.12) holds, indeed for any $|x| = R_0$ and $t \in (\underline{\tau}, t_c)$ we have that

$$\begin{aligned} u(t,x) &\geq \inf_{\substack{t \in (a t_c, t_c),\\ x \in B_{2R_0}(0)}} u(t,x) \geq \underline{C}\, a^{\frac{1}{2-p}}\, \frac{M_{R_0}}{R_o^d} \geq \frac{b^{p\,\beta} M_{R_0}^{p\,\beta}}{b_1^{\frac{p-1}{2-p}} (1-a)^{d\,\beta} t_c^{d\,\beta}} = \frac{(1-a)^{\frac{1}{2-p}}\, t_c^{\frac{1}{2-p}}}{\left[b_1 \frac{(1-a)^{\frac{p\,\beta}{p-1}} t_c^{\frac{p\,\beta}{p-1}}}{(b M_{R_0})^{(2-p)\frac{p\,\beta}{p-1}}} \right]^{\frac{p-1}{2-p}}} \\ &\geq \frac{(1-a)^{\frac{1}{2-p}}\, t_c^{\frac{1}{2-p}}}{\left[b_1 \frac{(1-a)^{\frac{p\,\beta}{p-1}} t_c^{\frac{p\,\beta}{p-1}}}{\left(b\, M_{R_0}\right)^{(2-p)\frac{p\,\beta}{p-1}}} + b_2 {R_0}^{\frac{p}{p-1}} \right]^{\frac{p-1}{2-p}}} = \sup_{\substack{t \in (a t_c, t_c),\\ |x| = R_0}} \mathcal{B}(t - \underline{\tau}, x; \underline{M}) . \end{aligned}$$

Recalling that $t_c = \kappa\, M_{R_0}^{2-p}\, R_0^{1/\beta}$ it is easy to show that inequality (5.2.13) is equivalent to the following one

$$b^{p\,\beta} \le \underline{C}\, b_1^{\frac{p-2}{2-p}}\, \kappa^{d\,\beta} a^{\frac{1}{2-p}}\, (1-a)^{d\,\beta}\ , \tag{5.2.14}$$

which is the condition we were looking for.

Compatibility of condition (5.2.11) *and* (5.2.14). Both conditions are satisfied by the following choice

$$a = \frac{1}{2} \ \text{ and } \ b^{p\,\beta} = b_1^{\frac{p-1}{2-p}}\, C\, \kappa^{d\beta} \left(\frac{1}{2}\right)^{\frac{p\,\beta}{2-p}} . \tag{5.2.15}$$

This concludes the proof of (5.2.9).

Case $t_0 \ge t_c$. Since inequality (5.2.7) holds for $t = t_c$ by comparison it holds for any $t \ge t_c$.

Case $0 < t_0 < t_c$. As already mentioned, we only need to prove inequality (5.2.7) at time t_0, the result will then follow by comparison. From the Benilan-Crandall-type estimate (5.1.8) follows tha for $0 < t_0 < t_c$

$$u(t_0, x) \ge u(t_c, x) \left(\frac{t_0}{t_c}\right)^{\frac{1}{2-p}} . \tag{5.2.16}$$

Inequality (5.2.9) holds under the choices of a and b as in (5.2.15). Using inequality (5.2.9) and (5.2.16) we get

$$\begin{aligned}
u(t_0, x) &\ge u(t_c, x) \left(\frac{t_0}{t_c}\right)^{\frac{1}{2-p}} \\
&\ge \frac{2^{-\frac{1}{2-p}}\, t_c^{\frac{1}{2-p}}}{\left[b_1 \dfrac{2^{-\frac{\beta\,p}{p-1}}\, t_c^{\frac{\beta\,p}{p-1}}}{\left(b\, M_{R_0}\right)^{(2-p)\frac{\beta\,p}{p-1}}} + b_2 |x|^{\frac{p}{p-1}} \right]^{\frac{p-1}{2-p}}} \left(\frac{t_0}{t_c}\right)^{\frac{1}{2-p}} \\
&= \frac{2^{-\frac{1}{2-p}}\, t_0^{\frac{1}{2-p}}}{\left[b_1 \dfrac{2^{-\sigma\vartheta} t_0^{\sigma\vartheta}}{\left(b\, M_{R_0} \left(\frac{t_0}{t_c}\right)^{\frac{1}{2-p}} \right)^{(2-p)\frac{\beta\,p}{p-1}}} + b_2 |x|^{\frac{p}{p-1}} \right]^{\frac{1}{2-p}}} \\
&= \mathcal{B}(t_0 - \underline{\tau}, x; \left(\frac{t_0}{t_c}\right)^{\frac{1}{2-p}} \underline{M}) ,
\end{aligned}$$

which is exactly (5.2.7) when $t_0 < t_c$. The proof is then concluded $\square$

5.2.3 Conservation of mass

We conclude this section with the proof of the conservation of mass for non negative solutions to (p-CP). This is a known result and it can be found in [148] and in [149]. The proof we give below is based only on some a priori estimates, we have decided to include it since we find it very simple (once the result of Lemma 5.4.4 is known).

Proposition 5.2.4. *Let $\frac{2N}{N+1} < p < 2$ and let $u(t,x)$ be the solution of Problem* (p-CP) *with initial datum $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. Then for any $t \ge 0$ we have that*

$$\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u_0(x)dx\,. \tag{5.2.17}$$

Proof. Using in the weak formulation (5.1.3) a cut-off function ϕ supported in B_{2R} such that $\phi = 1$ in B_R, we have that

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} u_0\phi(x)dx - \int_{\mathbb{R}^d} u(t,x)\phi(x)dx\Big| &\le C\frac{1}{R}\int_s^t\int_{B_{2R}(0)} |\nabla u|^{p-1}\\ &\le C\left(\frac{t-s}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{p}}\left[\int_{B_{4R}} u_0dx + \left(\frac{t}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}}\right]^{\frac{2(p-1)}{p}}\,.\end{aligned}$$

where in the middle step we have used Lemma 5.4.4. Taking the limit for $R \to \infty$ we have the assertion. $\square$

5.3 Initial data in $\mathcal{X}_p$. Global Harnack principle and uniform convergence in relative error

In this section we are going to prove the upper bound of inequality (5.1.2) for initial data in $\mathcal{X}_p$. As already observed, the space $\mathcal{X}_p$ plays a key role in the understanding of the Global Harnack Principle. Intuitively, function in $\mathcal{X}$ decay at ∞ as fast as the Barenblatt, however our intuition here is false. Indeed, it is possible to cook up some counterexamples for which such a statement is false. So it is somehow surprising that for such examples the GHP still holds.

5.3.1 Upper bound and global Harnack principle

As already observed in the Introduction Theorem 5.1.2 is divided in two parts: the upper bound and the lower bound of inequality (5.1.6). In this section we are going to discuss the upper bound.

Theorem 5.3.1. *Let u be the solution of Problem* (p-CP) *and let $0 \le u_0 \in \mathcal{X}_p$. Then for any $t_0 > 0$ there exist constants $\overline{\tau}$, $\overline{M}$ such that*

$$u(t,x) \le \mathcal{B}(t+\overline{\tau}, x; \overline{M}) \quad \text{for all } x \in \mathbb{R}^d\,,\ t \ge t_0\,, \tag{5.3.1}$$

where $\overline{\tau} = \overline{\tau}(t_0, \|u_0\|_{\mathcal{X}_p})$, $\overline{M} = \overline{M}(t_0, \overline{\tau}, \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)})$ and are given in (5.3.9) *and in* (5.3.10) *respectively.*

Proof of Theorem 5.3.1. We will prove first inequality (5.3.1) under the stronger hypothesis that there exists $R_0 > 0$ such that

$$u_0(x) \le A|x|^{-\frac{p}{2-p}} \quad \text{for all } |x| \ge R_0\,. \tag{5.3.2}$$

At the end of the proof we will show that such a decay condition is satisfied by a solution $u(t,x)$ of Problem (p-CP) if the initial data $u_0 \in \mathcal{X}_p$.

Let us fix a value $t_0 > 0$. Without loss of generality, we can assume that $\int_{\mathbb{R}^d} u_0(x)dx = 1$. It is sufficient to prove estimate (5.3.1) for time $t = t_0$, the result for larger times $t > t_0$ follows by comparison principle. We prove there exists a suitable choice of parameters $\overline{M}$ and $\overline{\tau}$ such that

$$u(t_0, x) \leq \mathcal{B}(t_0 + \overline{\tau}, x; \overline{M}), \quad \forall x \in \mathbb{R}^d. \tag{5.3.3}$$

The strategy is as follows: in view of the decay of the data (5.3.2), firstly, we determine sufficient conditions for the solution $u(t_0, x)$ to be bounded from above by a singular Barenblatt solution $\mathcal{U}$ for $|x| \geq R_0$. This is an upper barrier which meets the upper bound given by the smoothing effect (5.1.11) at some point $|x| = R_1$. Then we find the Barenblatt solution $\mathcal{B}(t_0 + \overline{\tau}, x; \overline{M})$ to be above the barrier $\mathcal{U}$ for all $|x| \geq R_1$. Inside the ball $\{|x| < R_1\}$ the comparison (5.3.3) follows by the monotonicity of $\mathcal{B}(t_0 + \overline{\tau}, x; \overline{M})$ in $|x|$.

Upper barrier outside a ball. We consider the singular Barenblatt solution starting at time S as it was previously introduced in (5.1.13)

$$\mathcal{U}(t, x; S) = b_2^{-\frac{p-1}{2-p}} (t + S)^{\frac{1}{2-p}} |x|^{-\frac{p}{2-p}}.$$

We continue by proving that under certain conditions, we can compare the initial data u_0 with an appropriate $\mathcal{U}(0, x; S)$.

If $R_0 = 0$, that is, estimate (5.3.2) holds in the whole space, then we choose

$$S \geq b_2^{p-1} A^{2-p} \tag{5.3.4}$$

then

$$u_0(x) \leq A|x|^{-\frac{p}{2-p}} \leq \mathcal{U}(0, x; S), \quad \forall x \in \mathbb{R}^d, x \neq 0.$$

It is known that $\mathcal{U}(t, x; S)$ is a supersolution of the equation in the domain $\{|x| > 0, t > 0\}$. Since $\mathcal{U}(t, 0; S) = +\infty > u(t, 0)$ for all $t > 0$, we conclude by the maximum principle that

$$\mathcal{U}(t, x; S) \geq u(t, x), \quad \forall x \in \mathbb{R}^d, t > 0.$$

In case $R_0 > 0$ we will use the singular Barenblatt solution defined in (5.1.14)

$$\mathcal{U}(t, x; B_1, S) = (t + S)^{\frac{1}{2-p}} \left[b_2 \, |x|^{\frac{p}{p-1}} - B_1 (t + S)^{\frac{\beta p}{p-1}} \right]^{-\frac{p-1}{2-p}}.$$

This function has a singularity on the surface $|x| = R_U(t) = (B_1/b_2)^{(p-1)/p}(t + S)^{\beta}$ where the denominator vanishes and it is a solution of the equation for $|x| > R_U(t)$. We want to prove that $u(t, x) \leq \mathcal{U}(t, x; B_1, S)$ in the parabolic domain $\{|x| > R_U(t); t > 0\}$. The comparison on the lateral boundary $\{|x| = R_U(t),\, t > 0\}$ is clearly satisfied since $\mathcal{U}(t, x; B_1, S) = +\infty$ here. It remain to find conditions on S and B_1 such the comparison at initial time $t = 0$ hold. Since u_0 satisfy (5.3.2) for $|x| \geq R_0$ and $\mathcal{U}(0, x; B_1, S)$ is defined for $|x| \geq R_U(0)$, then it is sufficient to choose the parameters of the barrier such that

$$R_0 = R_U(0) \quad \text{and} \quad A|x|^{-\frac{p}{2-p}} \leq S^{\frac{1}{2-p}} \left[b_2 \, |x|^{\frac{p}{p-1}} \right]^{-\frac{p-1}{2-p}}.$$

This holds when $R_0 = \left(\frac{B_1}{b_2}\right)^{\frac{p-1}{p}} S^{\beta}$ and $S \geq b_2^{p-1} A^{2-p}$. Thus we take S and B_1 such that

$$S \geq b_2^{p-1} A^{2-p}, \quad \text{and} \quad B_1 = b_2 \, R_0^{\frac{p}{p-1}} S^{-\frac{\beta p}{p-1}}. \tag{5.3.5}$$

We conclude that

$$u(t,x) \le \mathcal{U}(t,x;B_1,S), \quad \forall (t,x) \in \{|x| > R_U(t),\, t>0\}.$$

This estimate holds also when $R_0 = 0$ since in this case $R_U(0) = 0$ and then $B_1 = 0$.

Upper estimates in the whole space. We determine the point R_1 where the upper barrier $\mathcal{U}(t,x;B_1,S)$ meets the one given by the smoothing effect (5.1.11) at time $t = t_0$:

$$C_1 t_0^{-N\beta} = \mathcal{U}(t_0, R_1; B_1, S).$$

Simply computations show that the above inequality is satisfied by the following choice of R_1

$$b_2 R_1^{\frac{p}{p-1}} = \frac{t_0^{\frac{N\beta(2-p)}{p-1}} (t_0+S)^{\frac{1}{p-1}}}{C_1^{\frac{2-p}{p-1}}} + B_1 (t_0+S)^{\frac{\beta p}{p-1}}\,. \tag{5.3.6}$$

We observe that in view of the definition for B_1 given in (5.3.5) we have $R_0 \le R_1$. So far we have obtained a first upper bound for the solution $u(x,t)$ in the whole space:

$$u(t,x) \le \begin{cases} C_1 t_0^{-N\beta}, & \text{for } |x| \le R_1 \\ \mathcal{U}(t_0, x; B_1, S), & \text{for } |x| \ge R_1. \end{cases}$$

Finding the Bareblatt solution. We look for condition on $\overline{\tau}$ and $\overline{M}$ such that

$$\mathcal{B}(t_0+\overline{\tau}, x; \overline{M}) \ge \mathcal{U}(t_0, x; B_1, S)\,, \quad \text{for any } |x| \ge R_1. \tag{5.3.7}$$

We observe that the above condition is sufficient to obtain (5.3.3) since, by the monotonicity in $|x|$ of $\mathcal{B}$, one has that for $|x| \le R_1$:

$$u(t,x) \le C_1 t_0^{-N\beta} = \mathcal{U}(t_0, R_1; B_1, S) \le \mathcal{B}(Rt+\overline{\tau}, R_1; \overline{M}) \le \mathcal{B}(t+\overline{\tau}, x; \overline{M})\,.$$

Straightforward (but lengthy!) computations show that inequality (5.3.7) is equivalent to the following one

$$b_2 |x|^{\frac{p}{p-1}} \left[(t_0+S)^{-\frac{1}{p-1}} - (t_0+\overline{\tau})^{-\frac{1}{p-1}} \right] \ge b_1 \frac{(t_0+\overline{\tau})^{\frac{\beta p-1}{p-1}}}{\overline{M}^{(2-p)\frac{\beta p}{p-1}}} + B_1 (t_0+S)^{\frac{\beta p-1}{p-1}}\,, \quad \text{for any } |x| \ge R_1.$$

The left-hand-side of the above inequality is monotone increasing in $|x|$, therefore it is only necessary to verify it in $|x| = R_1$, namely $\overline{\tau}$ and $\overline{M}$ need to satisfy

$$b_2 R_1^{\frac{p}{p-1}} \left[(t_0+S)^{-\frac{1}{p-1}} - (t_0+\overline{\tau})^{-\frac{1}{p-1}} \right] \ge b_1 \frac{(t_0+\overline{\tau})^{\frac{\beta p-1}{p-1}}}{\overline{M}^{(2-p)\frac{\beta p}{p-1}}} + B_1 (t_0+S)^{\frac{\beta p-1}{p-1}}. \tag{5.3.8}$$

Using the definition of R_1 given by (5.3.6) and the one of B_1 given by (5.3.5) we find that condition (5.3.8) is equivalent to

$$\frac{t_0^{\frac{\beta p-1}{p-1}}}{C_1^{\frac{2-p}{p-1}}} (t_0+\overline{\tau})^{\frac{1}{p-1}} \ge b_1 \frac{(t_0+\overline{\tau})^{\frac{\beta p}{p-1}}}{\overline{M}^{(2-p)\frac{\beta p}{p-1}}} + b_2 \left(\frac{R_0}{S^\beta}\right)^{\frac{p}{p-1}} (t_0+S)^{\frac{\beta p}{p-1}} + \frac{t_0^{\frac{\beta p-1}{p-1}}}{C_1^{\frac{2-p}{p-1}}} (t_0+S)^{\frac{1}{p-1}}.$$

Such a condition is satisfied by the choices of

$$\overline{\tau} \geq \left[K \left(\frac{1}{S} + \frac{1}{t_0} \right)^{\frac{\beta p}{p-1}} t_0^{\frac{1}{p-1}} + 2(t_0+S)^{\frac{1}{p-1}} \right]^{p-1} - t_0 \quad \text{where} \quad K := 2b_2 C_1^{-\frac{2-p}{p-1}} R_0^{\frac{p}{p-1}} , \tag{5.3.9}$$

and

$$\overline{M}^{(2-p)\frac{\beta p}{p-1}} \geq 2b_1\, C_1^{-\frac{2-p}{p-1}} \left(\frac{t_0+\overline{\tau}}{t_0} \right)^{\frac{\beta p-1}{p-1}} . \tag{5.3.10}$$

It only remains to show that initial data in $\mathcal{X}_p$ will satisfy the decay condition (5.3.2).

Initial data in $\mathcal{X}_p$. Let $u_0 \in \mathcal{X}_p$, we will show that for any $t > 0$ there exists $A(t)$ such that

$$u(t,x) \leq \frac{A(t)}{|x|^{\frac{p}{2-p}}} , \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\} .$$

Let $x_0 \in \mathbb{R}^d$ and $R_0 = \frac{|x_0|}{4}$, by inequality (5.1.12) we have that

$$\begin{aligned} u(t,x_0) &\leq \frac{\kappa_1}{t^{d\beta}} \left(\int_{B_{2R_0}(x_0)} u_0(x)\,\mathrm{d}x \right)^{p\beta} + \kappa_2 \left(\frac{t}{R_0{}^p} \right)^{\frac{1}{2-p}} \\ &\leq \frac{\kappa_1}{t^{d\beta}} \left(\int_{B^c_{R_0}(x_0)} u_0(x)\,\mathrm{d}x \right)^{p\beta} + \kappa_2 \left(\frac{t}{R_0{}^p} \right)^{\frac{1}{2-p}} , \\ &\leq \frac{\kappa_1}{t^{d\beta}} \left(\frac{4^{\frac{1}{(2-p)\beta}} \|u_0\|_{\mathcal{X}_p}}{|x_0|^{\frac{1}{(2-p)\beta}}} \right)^{p\beta} + \kappa_2 \left(4^p \frac{t}{|x_0|^p} \right)^{\frac{1}{2-p}} = \frac{A(t)}{|x_0|^{\frac{p}{2-p}}} , \end{aligned}$$

where we have used the fact that $B_{2R_0}(x_0) \subset B^c_{R_0}(0)$ and the definition of the space $\mathcal{X}_p$. To fully obtain the result we just need to apply the precedent steps to the function $u(t_0,x)$. The proof is then complete. □

Proof of Theorem 5.1.2. The proof consists in a simple application of Theorem 5.2.2 and Theorem 5.3.1. □

5.3.2 Uniform convergence in relative error

The main goal of this subsection is to prove Theorem (5.1.4). We will split the proof in two steps: we will first prove Theorem (5.3.2), which gives us the convergence in relative error for data in $\mathcal{X}_p$. We consider this as a result on its own, since, as far as we know, it was not known before. Finally, we will show that if such a convergence take place, then the initial data $u_0 \in \mathcal{X}_p$. The proof of this second part is postponed to the end of the section.

Theorem 5.3.2. *Let u be the solution of Problem* (p-CP) *and let $0 \leq u_0 \in \mathcal{X}_p \setminus \{0\}$ and let $M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}$. Then the following limit holds*

$$\lim_{t\to\infty} \left\| \frac{u(t,x)}{\mathcal{B}(t,x;M)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \to 0 . \tag{5.3.11}$$

Proof of Theorem 5.3.2. We will prove that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\left\| \frac{u(t,x)}{\mathcal{B}(t,x;M)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \varepsilon\,, \quad \text{for all } t \geq t_\varepsilon\,. \tag{5.3.12}$$

We will need some previous results.

Claim 1. Let $u(t,x)$ be the solution of Problem (p-CP) and let $0 \leq u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ with $M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}$. Then the following limit holds

$$\lim_{t\to\infty} t^{d\beta} \|u(t,x) - \mathcal{B}(t,x;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0\,. \tag{5.3.13}$$

We have not found any reference to this result. Very similar results for the FDE/PME have been proven in [106] and in [107], however the exact statement of the above claim seems to be missing in the literature. In [147] the authors prove the above statement in the case $p > 2$ using the so called "4 step method" (see also [107] for a detailed account of this method), we claim that their prooof may be adapted to the case $\frac{2d}{d+1} < p < 2$ with some modifications.

From the above *Claim 1* the following *Claim 2* is easily proven

Claim 2. Let $u(t,x)$ be the solution of Problem (p-CP) and let $0 \leq u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ with $M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}$. Then for any $C > 0$ the following limit holds

$$\lim_{t\to\infty} \sup_{\{|x|\leq C\,t^\beta\}} \left| \frac{u(t,x)}{\mathcal{B}(t,x;M)} - 1 \right| = 0\,. \tag{5.3.14}$$

Proof of Claim 2. We argue by the following chain of inequalities

$$\sup_{\{|x|\leq C\,t^\beta\}} \left| \frac{u(t,x)}{\mathcal{B}(t,x;M)} - 1 \right| \leq \|u(t,x) - \mathcal{B}(t,x;M)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\, t^{d\,\beta} \left[\frac{b_1}{M^{(2-p)\frac{\beta\,p}{p-1}}} + b_2 C \right]^{\frac{p-1}{2-p}}\,,$$

thanks to *Claim 1*, the right-hand-side of the above converges to 0. This ends the proof of *Claim 2*.

The result of *Claim 2* is needed to control the convergence in *relative error* in domains of type $\{|x| \leq Ct^\beta\}$, while the GHP is needed to control the tails. We take advantage of the simple fact, already observed in [107, 118], that any Barenblatt solution $\mathcal{B}(t,x;M)$ behaves independently of the mass, for $|x| \sim \infty$, indeed

$$\mathcal{B}(t,x;M) \sim \frac{t^{\frac{1}{2-p}}}{b_2^{\frac{p-1}{2-p}}\,|x|^{\frac{p}{2-p}}}\,, \quad \text{for } |x| \to \infty\,. \tag{5.3.15}$$

Fix $\varepsilon > 0$ and $t_0 > 0$, by applying Theorem 5.1.2 we know that for any $t \geq t_0$ we have that

$$\mathcal{B}(t - \underline{\tau}, x; \underline{M}) \leq u(t,x) \leq \mathcal{B}(t + \overline{\tau}, x; \overline{M})\,,$$

for some $\underline{\tau}, \overline{\tau}, \underline{M}, \overline{M} > 0$. Using (5.3.15) it is possible to prove that in domains of the type $\{|x| \geq C\,t^\beta\}$, for $C > 0$ large enough, we have that

$$\left(1 - \frac{\varepsilon}{3}\right)\left(\frac{t - \underline{\tau}}{t}\right) \leq \frac{u(t,x)}{\mathcal{B}(t,x;M)} \leq \left(1 + \frac{\varepsilon}{3}\right)\left(\frac{t + \overline{\tau}}{t}\right)$$

We conclude that there exists $t'_\varepsilon > 0$ such that

$$1-\varepsilon \le \frac{u(t,x)}{\mathcal{B}(t,x;M)} \le 1+\varepsilon\,, \quad \text{for all } t \ge t'_\varepsilon\,, \quad \text{and } x \in \{|x| \ge C\,t\}\,. \tag{5.3.16}$$

To conclude the proof of the *Claim* we only need to prove an inequality similar to the previous one but for $(t,x) \in \{|x| \ge C\,t\}$. By (5.3.14) there exists $t''_\varepsilon > 0$ such that

$$\sup_{\{|x|\le 2C\,t^\beta\}} \left|\frac{u(t,x)}{\mathcal{B}(t,x;M)} - 1\right| \le \varepsilon\,, \quad \text{for all } t \ge t''_\varepsilon\,. \tag{5.3.17}$$

By combining (5.3.16) with (5.3.17) we obtain the proof of inequality (5.3.12). □

We can now complete the proof of Theorem 5.1.4.

Proof of Theorem 5.1.4. We have already seen that if $u_0 \in \mathcal{X}_p$ then $u(t)$ converges to the Barenblatt profile $\mathcal{B}(t,x;\|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)})$ in relative error, this is the result of Theorem 5.3.2. We only need to prove the converse. Assume that $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ is such that convergence in relative error holds, we will prove that $u_0 \in \mathcal{X}_p$. By convergence in relative error we have that there exists $t_0 > 0$ such that

$$u(t,x) \le 2\mathcal{B}(t,x;M)\,, \quad \text{for all } t \ge t_0\,.$$

By inequality (5.4.3) we have than that for any $R > 0$

$$R^{\frac{1}{\beta(2-p)}} \int_{B^c_R(0)} u_0(x)\,\mathrm{d}x \le \kappa_3 \left[2\,R^{\frac{1}{\beta(2-p)}} \int_{B_{R/2}(0)} \mathcal{B}(t_0,x;M) + t_0^{\frac{1}{2-p}}\right]\,.$$

Since $\mathcal{B}(t;M) \in \mathcal{X}_p$ for any $t > 0$, by taking the supremum in $R > 0$ in the above inequality, we conclude that $u_0 \in \mathcal{X}_p$. The proof is then complete. □

5.3.3 Decay of the gradient for radial data

It is widely known that the theory of the p-Laplace equation, $u_t = \Delta_p(u)$, has it counterparts in the various diffusive equations, for example the Porous Medium or Fast Diffusion Equation

$$u_t = \Delta u^m\,,$$

if the parameter $m > 1$ we are in the Porous Medium regime, while for $\frac{d-2}{d} < m < 1$ the above is called the Fast Diffusion Equation. The relation among the two equations is even clearer when we consider radial solutions, indeed there exists a transformation which maps radial solution of PME to radial solutions to the P-Lapacian Equation (PLE). In what follows we consider radial solutions defined on $\mathbb{R}^d$, d being the topological dimension of the ambient space. We will denote by $\overline{r} = |x|$ the coordinates in the FDE case and by $r = |x|$ in the PLE case. Assume that $\overline{u}(t,\overline{r})$ is a radial solution of the PME/FDE, then it satisfies

$$\overline{u}_t = \overline{r}^{1-\overline{n}}\,\frac{\partial}{\partial\overline{r}}\left(\overline{r}^{\overline{n}-1}\,|\overline{u}|^{m-1}\,\overline{u}_r\right)\,. \tag{5.3.18}$$

Similarly, a radial solution to the PLE $u(t,r)$ satisfies

$$u_t = r^{1-n}\,\frac{\partial}{\partial r}\left(r^{n-1}\,|u_r|^{p-2}\,u\right)\,. \tag{5.3.19}$$

Here we will consider both the parameter n and $\overline{n}$ to be continuous and the functions $u(t,r), \overline{u}(t,\overline{r}) : \mathbb{R}^d \to \mathbb{R}$. It doesn't make sense, in general, to consider such parameters as continuous quantities, however, in the radial case, this allows us to unveil some unexpected features.

In [130] the following radial equivalence has been proven

Theorem 5.3.3 (Theorem 1.2, [130]). *Suppose $2 < \overline{n} < \infty$. Then the radially symmetric solutions u and $\overline{u}$ of the FDE, resp. PLE, are related through the following transformation:*

$$u_r(t,r) = D\,\overline{r}^{\frac{2}{m+1}}\,\overline{u}(t,\overline{r})\,, \quad D = \left(\frac{(2m)^2}{m(m+1)^2}\right)^{\frac{1}{m-1}}\,, \tag{5.3.20}$$

where the correspondence of the parameters is

$$p = m+1 \quad n = \frac{(\overline{n}-2)(m+1)}{2m}\,, \tag{5.3.21}$$

and the independent variables are related by $r = \overline{r}^{\frac{2m}{m+1}}$.

In [130] the authors also analyze the case $0 < \overline{n} < 2$, however we have decided to not report their results here since we are not going to use them.

It may not be clear at a first glance but when the parameter $\overline{n}$ does not coincide with d, equation (5.3.18) is a Weighted Fast Diffusion Equation with Caffarelli-Kohn-Nirenberg weights. Indeed, we can write the right-hand-side of (5.3.18) in a more suggestive way

$$\overline{r}^{1-\overline{n}}\,\frac{\partial}{\partial \overline{r}}\left(\overline{r}^{\overline{n}-1}\,|\overline{u}|^{m-1}\,\overline{u}_{\overline{r}}\right) = |x|^{\gamma}\,\nabla.\left(|x|^{-\gamma}\,|\overline{u}|^{m-1}\,\nabla\overline{u}\right)\,, \tag{5.3.22}$$

where

$$\gamma = d - \overline{n}\,. \tag{5.3.23}$$

In the following result the aforementioned radial correspondence is key.

Theorem 5.3.4. *Let $u(t,x)$ be the solution of Problem* (p-CP) *with a smooth ($C^\infty(\mathbb{R}^d)$) radial initial data $u_0(|x|)$ and assume that there exists $A > 0$ and $R_0 > 0$ such that*

$$|\nabla u_0(r)| \le \frac{A}{r^{\frac{2}{2-p}}}\,, \quad \textit{for all } r \ge R_0\,. \tag{5.3.24}$$

Then, for any $t > 0$ there exists a constant $C = C(t,A) > 0$ such that,

$$|\nabla u(t,r)| \le \frac{C(t,A)}{(1+r)^{\frac{2}{2-p}}}\,, \quad \textit{for all } x \in \mathbb{R}^d\,. \tag{5.3.25}$$

Remark. We notice that the decay required in (5.3.24) is the one corresponding to $|\nabla\mathcal{B}|$. We also want to point out that condition (5.3.24) is not only-sufficient but necessary in the class of radial data. Indeed it is possible to construct counterexamples if that conditions is dropped.

Proof. We will use the aforementioned radial transformation and the Global Harnack Principle for solution to the Fast Diffusion Equation with Caffarelli-Kohn-Nirenberg weights proven in [150]. We will think of both $u(t,r) = u(t,|x|)$ and $\overline{u}(t,\overline{r}) = \overline{u}(t,|x|)$ as real valued functions defined on $\mathbb{R}^d$. As already mentioned above, the equation satisfied by $\overline{u}(t,|x|)$ is

$$\overline{u}_t(t,x) = |x|^{\gamma}\nabla.\left(|x|^{-\gamma}|\overline{u}|^{m-1}\,\nabla u\right)\,,$$

where

$$\gamma = d - \overline{n}\,, \quad \overline{n} = 2\frac{p-1}{p} + 2 \quad \text{and} \quad m = p-1\,.$$

To apply the known result to $\overline{u}(t,\overline{r})$ it is needed to verify that $\gamma < 0$ (namely $d < \overline{n}$) and that $m > m_c(\gamma) = \frac{\overline{n}-2}{\overline{n}}$. Indeed, we have that

$$d - \overline{n} = d - 2 - 2\,\frac{p-1}{p} d = \frac{(2-p)d - 2p}{p} < 0\,,$$

since $p > \frac{2d}{d+1}$. While the condition on m amounts to verify that

$$p > 2 - \frac{p}{p(1+d)-d} \quad \text{which is equivalent to} \quad p^2 - p\frac{3d+1}{d+1} + \frac{2d}{d+1} = (p-1)\left(p - \frac{2d}{d+1}\right) > 0\,,$$

again such a condition is satisfied since $p > \frac{2d}{d+1}$.

Finally, we observe that the solution $\overline{u}$ may be sign changing, however, this does not represent a problem in applying GHP, since we always have that

$$-\overline{u}_-(0,\overline{r}) \le \overline{u}(0,\overline{r}) \le \overline{u}_+(0,\overline{r})\,,$$

where $\overline{u}_-(0,\overline{r}) := \max\{0, |\overline{u}(0,\overline{r})|\}$ is the negative part of $\overline{u}$ and $\overline{u}_+(0,\overline{r}) := \max\{0, \overline{u}(0,\overline{r})\}$ is the positive part. Both $\overline{u}_-$ and $\overline{u}_+$ satisfy the following inequality

$$\overline{u}_-(0,\overline{r})\,, \overline{u}_+(0,\overline{r}) \lesssim \frac{\tilde{A}}{r^{\frac{2}{1-m}}}\,,$$

for some $\tilde{A} > 0$. Therefore, the GHP applies to $\overline{u}_-(t,\overline{r})$ ($\overline{u}_+(t,\overline{r})$ resp.), $\overline{u}_-(t,\overline{r})$ being the solution to the Cauchy Problem with initial data $\overline{u}_-(0,\overline{r})$ ($\overline{u}_+(0,\overline{r})$ resp.). By comparison, we deduce that the solution $|\overline{u}(t,\overline{r})|$ satisfies the following inequality

$$|\overline{u}(t,\overline{r})| \le \frac{A(t)}{\left(1+\overline{r}^2\right)^{\frac{1}{1-m}}}\,,$$

for some $A(t) > 0$. We obtain (5.3.25) as a consequence of the radial transformation (5.3.20). The proof is then complete. □

5.4 Some technical lemmata

Here we will write down some technical lemmata which were used in the previous section and whose proof we postponed to not interrupt the flow of ideas and theorems.

We first state a technical lemma, which will be widely used in what follows. Let us introduce the following notation, for any $0 \le r \le R$ we define the annulus $A(r,R)$ to be

$$A(r,R) := \{x \in \mathbb{R}^d : r \le |x| \le R\}\,. \tag{5.4.1}$$

The following lemma is a modification of [105, Lemma I.4.1, pag. 240].

Lemma 5.4.1. *Let* $\frac{2d}{d+1} < p < 2$ *and* u *be the solution of Problem* (p-CP). *There exists a constant* $\overline{\kappa} = \overline{\kappa}(d,p)$ *such that for any* $\varepsilon > 0$, *for any* $0 \le s \le t$ *and for any* $0 \le r \le R$ *the following inequality holds*

$$\begin{aligned}\int_s^t \int_{A(r,R)} |\nabla u|^{p-1} \psi^{p-1} dx d\tau \le \kappa\, (t-s)^{\frac{p-1}{p}} \left(\int_s^t \int_{A(r,R)} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)} |\nabla \psi|^p \right)^{\frac{p-1}{p}} \\ \times \left(\int_s^t \int_{A(r,R)} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)} \right)^{\frac{1}{p}} ,\end{aligned} \tag{5.4.2}$$

where $\psi(x)$ *is any smooth function supported in* $A(r,R)$.

The next two lemmas are of paramount importance in the above theory. The idea behind their use and proof is to quantify how fast the mass is transported locally and at infinity. We just recall that Lemma 5.4.2 is used to prove Theorem 5.1.4 while Lemma 5.4.3 is used in Proposition 5.2.4, namely, the conservation of mass.

Lemma 5.4.2. *Let* $\frac{2d}{d+1} < p < 2$ *and* u *be the solution of Problem* (p-CP). *There exists a constant* $\kappa_3 = \kappa_3(d,p)$ *such that for any* $R > 0$ *and for any* $T > 0$

$$\sup_{0 \le \tau \le T} \int_{B^c_{2R}(0)} u(x,\tau) dx \le \kappa_3 \left[\int_{B^c_R(0)} u(x,T) dx + \left(\frac{T}{R^{\frac{1}{\beta}}} \right)^{\frac{1}{2-p}} \right] . \tag{5.4.3}$$

Proof. In what follows, we will denote by B_R the ball of radius R centered at the origin $B_R = B_R(0)$. For any integer $k \ge 1$ define

$$R_k = 2R - R \sum_{i=1}^{k} 2^{-i} \quad \text{and} \quad \overline{R}_k = \frac{R_k + R_{k+1}}{2} \quad \text{both} \quad R_k\,, \overline{R}_k \searrow R\,. \tag{5.4.4}$$

Let us define for any $k \ge 0$ the function $x \to \xi_k(x)$ being a nonnegative smooth function such that

$$\xi_k = 0 \,\text{for}\, |x| \le \overline{R}_k\,, \quad \xi_k = 1 \,\text{for}\, |x| \ge R_k \quad \text{and} \quad |\nabla \xi_k| \le c \frac{2^{k+2}}{R}\,. \tag{5.4.5}$$

We notice that by a limit procedure, we are allowed to use ξ_k as test functions in (5.1.3). By testing the equation with $\phi = \xi_k$ we obtain, for any $0 \le s \le T$ that

$$\int_{B^c_{R_k}} u(s,x)\, \mathrm{d}x \le \int_{B^c_{\overline{R}_k}} u(T,x)\, \mathrm{d}x + c \frac{2^{k+2}}{R} \int_s^T \int_{A(\overline{R}_k, R_k)} |\nabla u|^{p-1}\, \mathrm{d}x \mathrm{d}\tau\,,$$

defining for any $k \ge 1$

$$M_k := \sup_{0 \le \tau \le T} \int_{B^c_{R_k}} u(\tau,x)\, \mathrm{d}x\,, \tag{5.4.6}$$

we have that for any $k \ge 1$

$$M_k \le \int_{B^c_R} u(T,x)\, \mathrm{d}x + c \frac{2^{k+2}}{R} \int_0^T \int_{A(\overline{R}_k, R_k)} |\nabla u|^{p-1}\, \mathrm{d}x \mathrm{d}\tau\,. \tag{5.4.7}$$

We can use inequality (5.4.2) of Lemma 5.4.1 with a test function ψ such that

$$\psi = 1 \text{ in } A(\overline{R}_k, R_k)\,,\ \psi = 0 \text{ in } A_k^c := A^c(R_{k+1}, R_k + (\overline{R}_k - R_{k+1}))\,,$$
$$\text{and} \quad |\nabla\psi| \le \frac{c}{\overline{R}_k - R_{k+1}} = \frac{c2^{k+2}}{R}\,,$$

with ε such that

$$\varepsilon = \left(\frac{T}{R^p}\right)^{\frac{1}{2-p}}\,,$$

to obtain

$$\begin{aligned}
\int_s^T \int_{A(\overline{R}_k,R_k)} |\nabla u|^{p-1}\,\mathrm{d}x\mathrm{d}\tau &\le \kappa \left(1 + \frac{T}{\varepsilon^{2-p}(\overline{R}_k - R_{k+1})}\right)^{\frac{p-1}{p}} \int_0^T \int_{A_k} (T-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)} \\
&\le C_1 2^{p(k+2)} \left[|A_k| \left(\frac{T}{R^p}\right)^{\frac{2(p-1)}{p(2-p)}} \int_0^T \frac{\mathrm{d}\tau}{(T-\tau)^{1-\frac{1}{p}}} + T^{\frac{1}{p}} \sup_{0\le\tau\le t} \int_{A_k} u^{\frac{2}{p}(p-1)}\,\mathrm{d}x \right] \\
&\le C_2 2^{p(k+2)} \left[R^N T^{\frac{1}{p}} \left(\frac{T}{R^p}\right)^{\frac{2(p-1)}{p(2-p)}} + T^{\frac{1}{p}} R^{N\frac{2-p}{p}} \sup_{0\le\tau\le T} \left(\int_{A_k} u\,\mathrm{d}x\right)^{\frac{2(p-1)}{p}} \right],
\end{aligned} \tag{5.4.8}$$

where we have used the fact that $|A_k| \le \kappa_d R^d$, where κ_d is a constant which depends only on the dimension d. Combining the above inequality with (5.4.8) and using the fact that

$$\int_{A_k} u(t,x)\,\mathrm{d}x \le \int_{B^c_{R_{k+1}}} u(t,x)\,\mathrm{d}x$$

we conclude that for any $k \ge 1$

$$M_k \le \int_{B_R^c} u(x,T)\,\mathrm{d}x + C_3 2^{pk} \left[\left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}} + \left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{p}} M_{k+1}^{\frac{2(p-1)}{p}} \right], \tag{5.4.9}$$

where C_3 depends only on d and on p. Fix $\delta \in (0,1)$ to be chosen later, by Young inequality we have that

$$C_3 2^{pk} \left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{p}} M_k^{\frac{2(p-1)}{p}} \le \delta M_{k+1} + C(d,p,\delta) 2^{\frac{p^2\,k}{(2-p)}} \left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}}\,,$$

where $C(d,p,\delta) = \left(C_3^p\,\delta^{2(p-1)}\right)^{\frac{1}{2-p}}$. Combining the above formula, with inequality (5.4.9) we obtain for any $k \ge 1$

$$M_k \le \delta\, M_{k+1} + C(d,p,\delta)\, 2^{\frac{p^2\,k}{(2-p)}} \left[\int_{B_R^c} u(x,T)\,\mathrm{d}x + \left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}} \right]. \tag{5.4.10}$$

Call $Z = \left[\int_{B_R^c} u(x,T)\,\mathrm{d}x + \left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}}\right]$, then we have the following iterative process

$$\begin{aligned}\sup_{0\le\tau\le T}\int_{B_{2R}^c} u(\tau,x)\,\mathrm{d}x \le M_1 &\le \delta\, M_2 + C(d,p,\delta)\, 2^{\frac{p^2}{(2-p)}} Z\\ &\le \delta^2\, M_3 + Z\, C(d,p,\delta)\left(2^{\frac{p^2}{(2-p)}} + 2^{\frac{p^2}{(2-p)}}\,\delta\right)\\ &\le \delta^3\, M_4 + Z\, C(d,p,\delta)\, 2^{\frac{p^2}{(2-p)}}\left(1+\delta+\delta^2\, 2^{\frac{2p^2}{(2-p)}}\right)\\ &\le \delta^k\, M_{k+1} + Z\, C(d,p,\delta)\, 2^{\frac{p^2}{(2-p)}}\sum_{i=0}^{k-1}(\delta\, 2^{\frac{2p^2}{(2-p)}})^i\,.\end{aligned} \tag{5.4.11}$$

Choosing $\delta = 2^{-\frac{2p^2}{(2-p)}-1}$ and taking the limit for $k\to\infty$ we obtain

$$\sup_{0\le\tau\le T}\int_{B_{2R}^c} u(x,\tau)\,\mathrm{d}x \le \kappa_3\left[\int_{B_R^c} u(x,T)\,\mathrm{d}x + \left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}}\right],$$

which is what we wanted to prove. □

As a Corollary of the method of the previous proposition one can prove the following one (see also [105, Lemma III3.1]).

Lemma 5.4.3. *Let* $\frac{2d}{d+1} < p < 2$ *and* u *be the solution of Problem* (p-CP)*. There exists a constant* $\kappa_4 = \kappa_4(d,p)$ *such that for any* $R>0$ *and for any* $T>0$

$$\sup_{0\le\tau\le T}\int_{B_R(0)} u(\tau,x)\,\mathrm{d}x \le \kappa_4\left[\int_{B_{2R}(0)} u(T,x)\,\mathrm{d}x + \left(\frac{T}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}}\right]. \tag{5.4.12}$$

The proof of the above lemma follows the line of the proof of Lemma 5.4.2, the only difference is that we shall implement the entire procedure in balls B_{R_k} instead of domains of type $B_{R_k}^c$.

Finally, we provide a proof of Lemma 5.4.4 which is used in the proof of conservation of mass.

Lemma 5.4.4. *Let* $\frac{2d}{d+1} < p < 2$ *and* u *be the solution of Problem* (p-CP)*. There exists a constant* $\kappa_5 = \kappa_5(d,p)$ *such that for any* $R>0$ *and for any* $0\le s\le t\le T$

$$\frac{1}{R}\int_s^t\int_{B_R(0)} |\nabla u|^{p-1}\,\mathrm{d}x\mathrm{d}\tau \le \kappa_5\left(\frac{\mathrm{t-s}}{\mathrm{R}^{\frac{1}{\beta}}}\right)^{\frac{1}{\mathrm{p}}}\left[\int_{\mathrm{B_{4R}}} \mathrm{u_0}\,\mathrm{dx} + \left(\frac{\mathrm{t}}{\mathrm{R}^{\frac{1}{\beta}}}\right)^{\frac{1}{2-\mathrm{p}}}\right]^{\frac{2(\mathrm{p}-1)}{\mathrm{p}}}. \tag{5.4.13}$$

Proof. We begin using inequality (5.4.2) of Lemma 5.4.1 with ψ a cut-off function such that $\psi=1$ on $B_R(0)$ and $\psi=0$ in $B_{2R}^c(0)$. We then obtain for any $\varepsilon = ((t-s)/R^p)^{\frac{1}{2-p}}$

$$\begin{aligned}\frac{1}{R}\int_s^t\int_{B_R(0)}|\nabla u|^{p-1} &\le \frac{C_1}{R}\int_s^t\int_{B_{2R}}(t-\tau)^{\frac{1}{p}-1}\left[u+\left(\frac{t-s}{R^p}\right)^{\frac{1}{2-p}}\right]^{\frac{2}{p}(p-1)}\mathrm{d}x\mathrm{d}\tau\\ &\le \frac{C_2(t-s)^{\frac{1}{p}}}{R}R^{\frac{d(2-p)}{p}}\left[\sup_{s\le\tau\le t}\int_{B_{2R}} u + \left(\frac{t-s}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}}\right]^{\frac{2(p-1)}{p}}\\ &\le C_3\left(\frac{t-s}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{p}}\left[\int_{B_{4R}} u_0\,\mathrm{d}x + \left(\frac{t}{R^{\frac{1}{\beta}}}\right)^{\frac{1}{2-p}}\right]^{\frac{2(p-1)}{p}},\end{aligned}$$

where in the last line we have used inequality (5.4.12) of Lemma 5.4.3. $\square$

Chapter 6

From the fast diffusion flow to stability in Gagliardo-Nirenberg Inequalities

The purpose of this chapter is to establish a new stability result for a special class of subcritical Gagliardo-Nirenberg inequalities. We develop a new strategy for studying the stability, in which the flow of the fast diffusion equation is used as a tool. This flow has regularity properties which allow us to reduce the problem to spectral issues of a properly linearized problem. The main novelty is that we can quantify all steps, including a global Harnack type result. As a consequence, we are able to establish explicit estimates for a stability region around the optimal functions in the Gagliardo-Nirenberg inequalities. Our estimates depend on the behaviour of the tail of the function in the stability result and of the tail of the initial datum in the case of the fast diffusion equation.

6.1 Introduction and main results

In the study of functional inequalities, the existence of an optimal function and its characterization is a standard problem of the Calculus of Variations: see for instance [151, 152]. Let us consider the Gagliardo-Nirenberg-Sobolev inequalities (GNS in what follows)

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathcal{C}_{\rm GN}\,\|f\|_{2p}\quad\forall\,f\in\mathcal{D}(\mathbb{R}^d)\,,\tag{6.1.1}$$

where $\mathcal{D}(\mathbb{R}^d)$ denotes the set of smooth functions on $\mathbb{R}^d$ with compact support. The exponent p is in the range $(1,+\infty)$ if $d=1$ or 2, and $p\in\big(1,d/(d-2)\big]$ if $d\ge3$. The exponent $\theta=\frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$ is determined by the scaling invariance. According to [12],

$$\mathsf{g}(x)=\big(1+|x|^2\big)^{-\frac{1}{p-1}}\quad\forall\,x\in\mathbb{R}^d$$

is an optimal function of (6.1.1), and the set of all optimal functions is the manifold of the functions $g_{\lambda,\mu,y}(x):=\mu\,\mathsf{g}\,((x-y)/\lambda)$ parametrized by $(\lambda,\mu,y)\in(0,+\infty)\times(0,+\infty)\times\mathbb{R}^d$.

Inequality (6.1.1) can also be written in non-scale invariant form as

$$\delta[f]:=(p-1)^2\,\|\nabla f\|_2^2+4\,\frac{d-p\,(d-2)}{p+1}\,\|f\|_{p+1}^{p+1}-2\,\mathcal{K}_{\rm GN}\,\|f\|_{2p}^{2p\,\frac{d\,(p-1)-2\,(p+1)}{d\,(p-1)-4\,p}}\ge0$$

and equality is again achieved by g. See [131, Section 4.1] for details on how $\mathcal{C}_{\rm GN}$ and $\mathcal{K}_{\rm GN}$ are related. Along this Chapter we will call δ the *deficit functional.*

In this chapter we study the *stability* properties of the GNS. The main question we want to address here is the following:

$$\textit{Assume that } \delta[f] \textit{ is small, in what sense, if any, is } f \textit{ close to } \mathsf{g}? \tag{Q}$$

The issue of *stability* of optimal functions in the Calculus of Variations started with the study of solitary waves obtained by minimization methods as in [132, 133, 134]. In recent years, the problem of finding stability estimates for the sharp inequalities both in analysis and geometry such as the isoperimetric inequality, the Brunn–Minkowski inequality, the Sobolev inequality, the logarithmic Sobolev inequality, etc, were intensively studied.

In the case of Sobolev inequality and GNS some early results have been obtained in bounded domains in [135, 136], but the result of G. Bianchi and H. Egnell for the critical Sobolev inequality [13] was immediately recognized as a major breakthrough, with the irritating drawback that the constant is still unknown. In the critical case $p = d/(d-2) = 2^*/2$, $d \geq 3$, G. Bianchi and H. Egnell proved in [13] the existence of a positive constant $\mathcal{C}$ such that

$$\frac{4}{(d-2)^2}\,\|\nabla f\|_2^2 - 2\,\mathcal{K}_{\rm GN}\,\|f\|_{2^*}^2 \geq \mathcal{C}\,\inf\|\nabla f - \nabla g\|_2^2\,,$$

where the infimum is taken over the $(d+2)$-dimensional manifold of the Aubin-Talenti functions. However, the existence of $\mathcal{C}$ is obtained by contradiction and no quantitative estimate of $\mathcal{C}$ has been obtained so far.

Several other results have been obtained since the result of G. Bianchi and H. Egnell. The stability of the Sobolev inequalities in the case $p \neq 2$ was proven by Cianchi, Fusco, Maggi and Pratelli in [14] and more recently by Figalli and Neumayer in [15]. For GNS the first stability results (to the best of our knowledge) is due to Figalli and Carlen in [16] and to Dolbeault and Toscani in [17].

Our goal is to establish a *quantitative* analogue of the estimate of G. Bianchi and H. Egnell in the subcritical range $p \in (1, 2^*/2)$. More specifically, we aim at proving that $\delta[f]$ controls a distance to the function g under some suitable assumptions. Here we devise an entirely new strategy based on a nonlinear flow and fine regularity properties which allows us to relate in a quantitative way a nonlinear regime with an asymptotic regime and a properly linearized problem. We apply our method to an entire family of subcritical Gagliardo-Nirenberg-Sobolev inequalities. Our approach breaks in the critical case, that is, in the case of Sobolev's inequality considered by Bianchi and Egnell, but at least sheds a new light on a quantitative stability theory in functional interpolation inequalities.

Before stating our main result, we need to introduce the *relative entropy functional*, let $d \geq 3$ and $p \in \big(1, d/(d-2)\big]$,

$$\mathcal{E}[f] = \left(\frac{2p}{1-p}\right)\int_{\mathbb{R}^d}\left(|f|^{p+1} - \mathsf{g}^{p+1} - \frac{1+p}{2p}\mathsf{g}^{1-p}(|f|^{2p} - \mathsf{g}^{2p})\right)dx\,, \tag{6.1.2}$$

it may not appear obvious but $\mathcal{E}$ is a positive functional. Indeed, $\mathcal{E}$ is naturally associated to the fast diffusion flow and in the next section the relation between δ and $\mathcal{E}$ will be clarified. It is interesting to notice that if $\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} = \|\mathsf{g}\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}$ then, by the Csiszár-Kullback inequality, the entropy controls the L^{2p} distance between f and g, namely there exists a constant $C_p > 0$ such that

$$\|f - \mathsf{g}\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p} \leq C_p\,\mathcal{E}[f]^{\frac{1}{2}}\,,$$

for further information see [17, 137, 138].

Let us denote $\mathrm{W}^{1,2}(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ that have a square-integrable distributional gradient. We are finally in the position of stating our main result.

Theorem 6.1.1. *Let $d \geq 3$ and and $p \in \big(1, d/(d-2)\big)$. Let $f \in \mathrm{W}^{1,2}(\mathbb{R}^d)$ such that $\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} = \|\mathsf{g}\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}$ and assume that*

$$\int_{\mathbb{R}^d} x f^{2p}\, dx = 0$$

and that for some $A, B > 0$ we also have

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} f^{2p}\, dx \leq A < \infty \qquad \text{and} \qquad \mathcal{E}[f] \leq B < \infty\,.$$

Then there exists a constant $\mathcal{C} > 0$ depending only on d, p and A, B such that

$$\delta[f] \geq \mathcal{C}\, \mathcal{E}[f] \tag{6.1.3}$$

Remark 6.1.2. At a first glance it may appear unnatural to consider the entropy of the function f^{2p}. However, it will be clear later, in section 6.1.1 that f^{2p} is nothing else then the density (or solution) of the Fast Diffusion Flow, so it makes sense to consider this power of the function f.

The reader will also recognize the tail condition in the defition of the space $\mathcal{X}$ considered in the previous chapters and now restated as

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} f^{2p}\, dx \leq A\,. \tag{6.1.4}$$

Once again, in what follows, f^{2p} will be the density of the Fast Diffusion Flow, since we will use the Global Harnack Principle proved in chapter 4 it is natural to ask f^{2p} to belong the space $\mathcal{X}$.

Finally, let us clearly state that (6.1.4) cannot be dropped and it is key in our method.

As a consequence of Theorem 6.1.4 we obtain the following Bianchi-Egnell type inequality.

Corollary 6.1.3. *Under the assumptions of Theorem 6.1.4 assume further that*

$$\|f^{2p}\,|x|^2\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|\mathsf{g}^{2p}\,|x|^2\|_{\mathrm{L}^1(\mathbb{R}^d)} \qquad \text{and} \qquad \delta[f] \leq 1$$

Then there exists a constant $\mathcal{C}_2 > 0$ which depends only on d, p and on A of Theorem 6.1.4 such that

$$\delta[f] \geq \mathcal{C}_2\, \frac{\|\nabla f - \nabla g\|^8_{\mathrm{L}^2(\mathbb{R}^d)}}{\Big(1 + \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}\Big)^4}\,. \tag{6.1.5}$$

Both Theorem 6.1.1 and Corollary 6.1.3 answer to the question Q, however the results are quite different. Inequality (6.1.3) of Theorem 6.1.1 affirms that the deficit controls the L^{2p} distance from f to g, while inequality (6.1.5) proves stability at the level of gradients which, as often mentioned in the literature, is the strongest possible norm in this context.

6.1.1 The deficit functional as the entropy-entropy production inequality

In order to study the GNS interpolation inequalities let us consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{6.1.6}$$

with an exponent m in the range $m_1 < m < 1$, $m_1 := (d-1)/d$. We recall that in this range conservation of mass holds. As we already mentioned along this text, equation (6.1.6) admits a family of self-similar solutions given by

$$B(t,x) = \frac{\mu^d}{R(t)^d}\,\mathcal{B}\left(\frac{\mu\,x}{R(t)}\right)$$

where $\mathcal{B}$ is the Barenblatt profile

$$\mathcal{B}(x) = \big(1+|x|^2\big)^{\frac{1}{m-1}} \quad \forall\,x\in\mathbb{R}^d$$

with mass $\mathcal{M}$, namely

$$\mathcal{M} = \int_{\mathbb{R}^d} \mathcal{B}(x)\,dx\,,$$

and where $\mu>0$ is such that

$$\mu^{2-d\,(1-m)} = \frac{1-m}{2\,m} \tag{6.1.7}$$

and

$$R(t) = (1+\alpha\,t)^{1/\alpha}\,, \quad \alpha = d\,(m-m_c)\,, \quad m_c = (d-2)/d\,. \tag{6.1.8}$$

The *free energy* (or *relative entropy*) and the *relative Fisher information* are defined respectively by

$$\mathcal{F}[u] := \frac{1}{m-1}\int_{\mathbb{R}^d}\big(u^m-\mathcal{B}^m-m\,\mathcal{B}^{m-1}(u-\mathcal{B})\big)\,dx \tag{6.1.9}$$

and

$$\mathcal{I}[u] := \frac{m}{1-m}\int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1}-\nabla\mathcal{B}^{m-1}\right|^2 dx\,.$$

We know from [12] that

$$\mathcal{F}[u] \le \frac{1}{4}\,\mathcal{I}[u]\,. \tag{6.1.10}$$

Indeed, again from [12], we know that the GNS inequality, or equivalently the fact that $\delta[f]\ge 0$, is indeed equivalent to inequality (6.1.10). Let $f\in \mathrm{W}^{1,2}(\mathbb{R}^d)$, $p\in\big(1,\frac{d-2}{d}\big)$ and $m\in(m_1,1)$ if

$$p = \frac{1}{2m-1} \qquad \text{and} \qquad u = |f|^{2p}\,, \tag{6.1.11}$$

then we have the following identity

$$\frac{p+1}{4\,(p-1)}\delta[f] = \mathcal{I}[u]-4\mathcal{F}[u]\,.$$

The relation (6.1.11) gives the following relation among $\mathcal{B}$ and g

$$\mathcal{B}(x) = \mathsf{g}^{2p}(x)\,.$$

Furthermore, again under the assumption of identities (6.1.11) we have that

$$\mathcal{E}[f] = \mathcal{F}[u]$$

where $\mathcal{E}$ is the functional defined in (6.1.2) while $\mathcal{F}$ is the one defined in (6.1.9). See [12] and [131] for a global overview.

With the notation of this section Theorem 6.1.1 can be restated as

Theorem 6.1.4. *Assume that* $m \in (m_1, 1)$ *and* $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d, dx)$ *with* $\mathcal{F}[u_0] < \infty$ *and* $\mathcal{I}[u_0] < \infty$. *Suppose also that*

$$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} \mathcal{B}\,dx \qquad \textit{and} \qquad \int_{\mathbb{R}^d} x_i\,u_0\,dx = \int_{\mathbb{R}^d} x_i\,\mathcal{B}\,dx \qquad \textit{for any } i = 1\,,\cdots,d\,.$$

and assume that for some $A, B > 0$ *we also have*

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{2\,(1-m)}} \int_{|x|>r} u_0\,dx \le A < \infty \qquad \textit{and} \qquad \mathcal{F}[u_0] \le B < \infty\,.$$

Then there exists a constant $\mathcal{C}_1 > 0$ *depending only on* d, m *and* A, B *such that*

$$\mathcal{I}[u_0] - 4\mathcal{F}[u_0] \ge \mathcal{C}_1\,\mathcal{F}[u_0] \tag{6.1.12}$$

6.1.2 Fast diffusion flow in self-similar variables

By the *self-similar change of variables* we mean

$$u(t,x) = \frac{\mu^d}{R(t)^d}\,v\,(\tau,y) \quad \text{where} \quad \tau = \frac{1}{2}\,\log R(t)\,,\ y = \frac{\mu\,x}{R(t)} \tag{6.1.13}$$

where μ and R are given respectively by (6.1.7) and (6.1.8), a solution of (6.1.6) with initial datum u_0 is transformed into a solution to

$$\frac{\partial v}{\partial \tau} + \nabla\left(v\,\nabla v^{m-1}\right) = 2\,\nabla\cdot(y\,v) \tag{6.1.14}$$

with same initial datum u_0. A key feature of (6.1.13) is that the self-similar solution $B(t,\cdot)$ is changed into the stationary solution $\mathcal{B}$. The description of the so-called *intermediate asymptotics*, that is, the large time behaviour of the solution u of (6.1.6) is now reduced to the convergence of the solution v to $\mathcal{B}$ as $\tau \to +\infty$, according to, *e.g.*, [106, 12]. Let us simply recall that the global existence of a non-negative solution corresponding to an initial datum $u_0 \ge 0$ has been established in [99] for any $m \in (m_1, 1)$. Much more is known and we refer to [4] for a global overview.

We know from [12] that the Gagliardo-Nirenberg inequality (in non-scale invariant form) determines an exponential rate of convergence of the solutions to the fast diffusion equation (6.1.6) when it is measured in relative entropy (relative with respect to the optimal Barenblatt functions). The optimal exponential rate is moreover equivalent to the knowledge of the optimal constant in the inequality. Indeed, if v solves (6.1.14) with initial datum u_0, we have

$$\frac{d}{d\tau}\mathcal{F}[v(\tau,\cdot)] = -\,\mathcal{I}[v] \le -\,4\,\mathcal{F}[v(\tau,\cdot)]\,, \tag{6.1.15}$$

which shows that

$$\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[u_0]\,e^{-4\,\tau} \quad \forall\,\tau \ge 0\,.$$

6.1.3 Idea of the proof of Theorem 6.1.4

In [128] the authors prove that if v is close to $\mathcal{B}$ in uniform relative error then an improved version of the entropy-entropy production holds. More precisely, assume that v solves (6.1.14) and satisfies, for a small enough $0 < \varepsilon < 1$ and for some $\tau_\varepsilon > 0$, the following assumption

$$\sup_{x\in\mathbb{R}^d} \left|\frac{v(\tau)}{\mathcal{B}} - 1\right| < \varepsilon \qquad \text{for any } \tau \ge \tau_\varepsilon\,, \tag{6.1.16}$$

then there exists a constant $\mathcal{C} = \mathcal{C}(\varepsilon) > 4$ such that

$$\mathcal{C}(\varepsilon)\,\mathcal{F}[v(\tau)] \le \mathcal{I}[v(\tau)] \qquad \text{for any } \tau \ge \tau_\varepsilon\,. \tag{6.1.17}$$

However, for solution to (6.1.14) inequality (6.1.16) holds for large times but not for initial data. Our main discovery is that it is the quotient

$$\frac{\mathcal{I}[v_0]}{4\mathcal{F}[v_0]}$$

where v_0 is the initial data, that determines how fast the solution $v(\tau)$ approaches $\mathcal{B}$. Let us fix some details. Let us consider the class of non-negative functions v such that

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{2\,(1-m)}} \int_{|x|>r} v\,dx \le A < \infty \tag{H$_A$}$$

for some positive parameter A and let us define $\mathcal{X}$ as

$$\mathcal{X} = \{v \in \mathrm{L}^1(\mathbb{R}^d) : v \ge 0, v \text{ satisfies } (\mathrm{H}_A)\}\,. \tag{6.1.18}$$

In chapter 4 we have proven that initial data in $\mathcal{X}$ produce solutions to (6.1.14) which converge uniformly in relative error to the Barenblatt profile $\mathcal{B}$ (once the mass of the initial data is properly fixed). For any $0 < \varepsilon < 1$ let us define

$$T_\star^\varepsilon\,(v_0) := \inf\{T \ge 0 : \left\|\frac{v(t)}{\mathcal{B}} - 1\right\|_{L^\infty(\mathbb{R}^d)} < \varepsilon \quad \text{for any} \quad t \ge T\}\,,$$

and consider the set

$$\mathcal{V}_\varepsilon^N := \{v_0 \in \mathcal{X} : T_\star^\varepsilon(v_0) \le N\}\,.$$

We shall prove that functions $v \in \mathcal{V}_\varepsilon^N$ satisfy an improved entropy-entropy production inequality, namely that

$$\mathcal{I}[v] - 4\mathcal{F}[v] \ge \mathcal{C}(\varepsilon, N)\,\mathcal{F}[v]\,, \tag{6.1.19}$$

where

$$\mathcal{C}(\varepsilon, N) = 4\frac{e^{-8N}}{1-e^{-8N}}\,\min\{1\,, \left(\frac{\mathcal{C}_\infty(\varepsilon)}{4} - 1\right)\left(e^{4N} - 1\right)\} > 0\,. \tag{6.1.20}$$

Our proof is a based on a dichotomy argument on the value of the quotient

$$\frac{\mathcal{I}[v_0]}{4\mathcal{F}[v_0]}$$

for functions $v_0 \in \mathcal{V}_\varepsilon^N$.

Let us explain how this chapter is organized. In section 6.2 we make use of the Global Harnack Principle to quantify the time $T_\star^\varepsilon\,(v_0)$ in terms of the initial mass and the parameter A of (H_A). In section 6.3 we first prove inequality (6.1.17) under the assumption (6.1.16), this is the result of Proposition 6.3.1. Later in Proposition 6.3.6 we prove inequality (6.1.19) for all data in $\mathcal{V}_\varepsilon^N$. Finally, the proof of Theorem 6.1.4 consists in finding sufficient conditions on the initial data v_0 (a bound on the entropy $\mathcal{F}[v_0]$ and on the parameter A) such that $v_0 \in \mathcal{V}_\varepsilon^N$ for a fixed N and ε.

6.1.4 Final comments and notation

In what follows we shall often make use of the self-similar change of variables and, with an abuse of notation and language, often refer to the Fast Diffusion Flow as either the flow given by the equation (6.1.6) or by (6.1.14), however, we believe that no confusion will arise.

In section (6.2) we shall make use of a Barenblatt profile with a different masses. Let us define

$$B(t\pm\tau,x;M)=\frac{(t\pm\tau)^{\frac{1}{1-m}}}{\left[b_0\frac{(t\pm\tau)^{\frac{2}{\alpha}}}{M^{(1-m)\frac{2}{\alpha}}}+b_1|x|^2\right]^{\frac{1}{1-m}}}\,, \tag{6.1.21}$$

where M is the mass of $B(t\pm\tau,x;M)$, b_0,b_1 are constants which depend on d,m and $\pm\tau$ is a translation in time. We recall that, by the semigroup property, the function $B(t\pm\tau,x;M)$ is a solution to (6.1.6) whenever $t\pm\tau>0$.

6.2 Global Harnack principle and uniform relative convergence: a quantitative approach

In Chapter 4 we have shown that any solution u of (6.1.6), whose initial data is in $\mathcal{X}$, converges to a suitable Barenblatt profile in uniform relative error, namely that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^d}\left|\frac{u(t,x)}{B(t,x)}-1\right|=0\,.$$

The validity of the above limit implies that for any $0<\varepsilon<1/2$ there exists for $t_\star>0$, which depends on $\varepsilon>0$ and some others parameters, such that

$$\sup_{x\in\mathbb{R}^d}\left|\frac{u(t,x)}{B(t,x)}-1\right|<\varepsilon\,. \tag{6.2.1}$$

Through all this section we shall assume that

$$\int_{\mathbb{R}^d}u_0\,dx=\int_{\mathbb{R}^d}\mathcal{B}\,dx\,.$$

Our main goal in this section would be to quantify and give an explicit expression to $t_\star$. We will prove the following theorem.

Theorem 6.2.1. *Assume that $m\in(m_1,1)$, $0<\varepsilon<1/2$ and $A>0$ are given. There exists a nonnegative constant $t_\star$ such that, if u is a solution of (6.1.6) with non-negative initial datum $u_0\in\mathrm{L}^1(\mathbb{R}^d)$ satisfying (H_A), then*

$$\sup_{x\in\mathbb{R}^d}\left|\frac{u(t,x)}{B(t,x)}-1\right|\le\varepsilon\quad\forall\,t\ge t_\star\,.$$

The expression of $t_\star$ is given at the end of the proof.

Once again our main tool will be the *Global Harnack Principle* (GHP in what follows). We just recall to the reader that we mean by GHP the following property: fix a $t_0>$ then there exists $\overline{\tau},\underline{\tau}>0$ and $\overline{M},\underline{M}>0$ such that we have

$$B(t-\underline{\tau},x;\underline{M})\le u(t,x)\le B(t+\overline{\tau},x;\overline{M})\,,\qquad\text{for any}\quad x\in\mathbb{R}^d\,,t\ge t_0\,, \tag{6.2.2}$$

where $B(t-\underline{\tau}, x; \underline{M})$ and $B(t+\overline{\tau}, x; \overline{M})$ are defined in (6.1.21). We have proven in chapter 4 that the above property holds for solution to (6.1.6) whose initial data belong to $\mathcal{X}$.

Let us give the idea of the proof. The GHP implies a control on the tails, namely the validity of inequality (6.2.1) in sets of the form $|x| \geq C\,t^\alpha$. However, this would be not enough. Indeed, to obtain (6.2.1) in sets of the form $|x| \leq Ct^\alpha$ we shall proceed in a different way. As was already done in Chapter 4, we will use regularity theory to estimate the C^ν seminorm of the difference $u(t,x) - B(t,x)$. Then, by means of an interpolation inequality, we shall prove that the entropy functional $\mathcal{F}[u]$ controls the L^∞ norm of the difference $u(t,x) - B(t,x)$ on sets of the form $|x| \leq Ct^\alpha$ which will be enough to show (6.2.1) in those sets. Most of the computations will be done in self-similar variables.

Our goal in this chapter is to give quantitative estimates, possibly with explicit expression in terms of m, d or in terms of some known quantities. So we shall give explicit expression, when possible, of the main constants which appear in the computations.

6.2.1 Quantitative upper bound of inequality (6.2.2)

Inequality (6.2.2) has already been proven in Chapter 4. However, since in what follows it will be easier to take track of the involved quantities, here we will state again the results of Chapter 4 with full details and with explicit constants. The upper bound of inequality (6.2.2) is given in the following proposition.

Proposition 6.2.2. *Assume that $m \in (m_1, 1)$ and $A > 0$. There exist nonnegative constants $\overline{M}, \overline{\tau}$ such that, if $u(t,x)$ is a solution of (6.1.6) with non-negative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying* (H_A)*, and if*

$$t_0 = \max\left\{2^{\frac{\alpha(14-(1-m))}{2}} \left(\frac{\kappa_1}{\kappa_2}\right)^{\frac{\alpha(1-m)}{2}} A^{1-m},\ \alpha^{-1}\left(\frac{1-m}{2^8\,m}\,(2\,\kappa_2)^{1-m} - \alpha\right)^{-1}\right\} \tag{6.2.3}$$

then

$$u(t,x) \leq B(t+\bar{t}-\frac{1}{\alpha}, x; \overline{M}) = \frac{\left(t+\bar{t}\right)^{\frac{1}{1-m}}}{\left[\frac{\mathcal{M}^{\frac{2(1-m)}{\alpha}}}{\overline{M}^{\frac{2(1-m)}{\alpha}}}\frac{\left(t+\bar{t}\right)^{\frac{2}{\alpha}}}{\mu^{d(1-m)}\alpha^{d(m-1)}} + \frac{1-m}{2m}\frac{|x|^2}{\alpha}\right]^{\frac{1}{1-m}}} \quad \forall\, t \geq t_0\,. \tag{6.2.4}$$

Remark 6.2.3. The condition $t_0 \geq 2^{\frac{\alpha(14-(1-m))}{2}}\left(\frac{\kappa_1}{\kappa_2}\right)^{\frac{\alpha(1-m)}{2}} A^{1-m}$ in Proposition 6.2.2 is not a necessary one, indeed a more general version of this result holds true and it is proven in [110, Thm 1.5]. However we believe that this weaker result give us a clearer view of the value of the parameters involved and is more suitable for our purposes. The condition $t_0 \geq \alpha^{-1}\left(\frac{1-m}{2^8\,m}\,(2\,\kappa_2)^{1-m} - \alpha\right)^{-1}$ is a technical one and its role will be clear in Section 6.2.4. The reader may observe that (6.2.6) amounts to $\frac{1-m}{2^8\,m}\,(2\,\kappa_2)^{1-m} - \alpha > 0$.

In the expression of t_0 given in (6.2.3) the constants κ_1 and κ_2 appear. The reader should know that those constants come from the following lemma.

Lemma 1. *Under the Assumptions of Theorem 6.2.1, there are two positive constants κ_1 and κ_2 such that, if any solution u of (6.1.6) with a nonnegative initial datum $u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies the* *pointwise parabolic estimate*

$$\sup_{y\in B_{R/2}(x)} u(t,y) \le \frac{\kappa_1}{t^{d/\alpha}}\left(\int_{B_R(x)} u_0(y)\,dy\right)^{2/\alpha} + \kappa_2\left(\frac{t}{R^2}\right)^{\frac{1}{1-m}} \qquad \forall\,(t,R)\in(0,+\infty)^2\,, \tag{6.2.5}$$

where

$$\kappa_1 = 2^{\frac{6}{\alpha}}18^{\frac{d+2}{\alpha}}(4\mathcal{S})^{\frac{d}{\alpha}}\,\frac{4\pi^4}{9}\,\frac{e^{\frac{4d(d+2)}{\alpha(4-d(1-m))}\sum_{j=1}^{\infty}\left(\frac{d}{d+2}\right)^j\log j}}{2^{\frac{2+d(1-m+m_c)}{\alpha}}-1}\,,$$

$$\kappa_2 = \frac{\kappa_1}{2^{\frac{6}{\alpha}}18^{\frac{d+2}{\alpha}}}\left[4\omega_d + 32\left(\frac{3}{2}\right)^{\alpha}\frac{2^{\frac{m+7}{1-m}}}{2-m}\left(d+\frac{3\cdot 2^6}{1-m}\right)^{\frac{1}{1-m}}\omega_d^{\frac{2}{d(1-m)}-m+m_c}\right]^{\frac{2}{\alpha}}\,,$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ *and the constants* $\mathcal{S}$ *can be estimated in terms of the* Γ *function.*

The above lemma is actually one of our main results of the first part of this thesis. Indeed, for the interested reader we refer to Chapter 1 for the proof. There we keep track of the constants and give explicit expressions.

Remark 6.2.4. Even if κ_1,κ_2 are explicit, understanding their value is not straightforward. Nevertheless we take advantage of the fact that they are used in an upper bound therefore we may assume that, for example, they are both bigger than a certain numerical value. A technical assumption will be that

$$\kappa_2^{1-m} > \alpha\,\frac{2^8\,m}{1-m}\,2^{m-1}\,. \tag{6.2.6}$$

6.2.2 Quantitative lower bound of inequality (6.2.2)

Here we give the details of the lower part of inequality (6.2.2). In [110, Theorem 1.1] the authors prove a lower estimate for solution to (6.1.6) which, in our setting, can be restated as follows. In part I of this thesis we have proven a very similar result, however, it is easier to keep track of the main constants if we use [110, Theorem 1.1].

Lemma 2. *Assume that* $m\in(m_1,1)$*. Then there exist non-negative constants* $\kappa,\kappa_\star$*, such that, if* $u(t,x)$ *is a solution to* (6.1.6) *with non-negative initial datum* $u_0\in\mathrm{L}^1(\mathbb{R}^d)$*, and let* $R_\star>0$ *be such that* $\int_{|x|\le R_\star} u_0\,dx = \mathcal{M}/2$*, then*

$$\inf_{B_{R_\star}(0)} u(t,x) \ge \kappa\left(\frac{t}{R_\star^2}\right)^{\frac{1}{1-m}} \qquad \forall\, t\in[0,t_1]\,, \quad \textit{where} \quad t_1 = \kappa_\star\,\mathcal{M}^{1-m}\,R_\star^{\alpha}\,, \tag{6.2.7}$$

where

$$\kappa_\star = \frac{2^{m+\frac{\alpha}{d(1-m)}((d-1)(1-m+4))}}{(1-m)^{\frac{\alpha}{(1-m)d}}}\left[\max\left\{1,\kappa_1\,\omega_d\right\}\right]^{\frac{\alpha}{d}}\left[\max\left\{1,4\,d\,2^d\,\omega_d\right\}\right]^{\frac{\alpha}{d}}$$

$$\kappa = (d(1-m))^{\frac{2}{\alpha}-1}\,\frac{2^{\frac{\alpha-d-4}{\alpha}}}{\kappa_\star^{\frac{1}{1-m}}}\,\frac{\left(2^{\frac{d-1}{d}}-1\right)^d}{2^{2d-1}+2^d}\left[\max\left\{1,\kappa_1\,\omega_d\right\}\right]^{-1}\left[\max\left\{1,4\,d\,2^d\,\omega_d\right\}\right]^{-1-\frac{2}{\alpha}}$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ *and* κ_1 *is as in Lemma 1.*

Remark 6.2.5. Under the assumption (H_A) we are able to provide an upper bound on the time t_1 defined in (6.2.7). Since, by defintion of $R_\star$ we have that

$$\int_{|x|\le R_\star} u_0\,dx = \int_{|x|\ge R_\star} u_0\,dx = \mathcal{M}/2\,,$$

we can argue as follows

$$t_1^{\frac{1}{1-m}} = \kappa_\star^{\frac{1}{1-m}}\,\mathcal{M}\,R_\star^{\frac{\alpha}{1-m}} = 2^{\frac{1}{1-m}}\,\kappa_\star^{\frac{1}{1-m}}\;R_\star^{\frac{\alpha}{1-m}}\int_{|x|\ge R_\star} u_0\,dx \le 2^{\frac{1}{1-m}}\,\kappa_\star^{\frac{1}{1-m}}\,A \tag{6.2.8}$$

where in the last inequality we have used (H_A).

We do not give a proof of Lemma 2 since it can be found in [110, Theorem 1.1] or in Chapter 3. However, we want to stress that the value of the constants $\kappa, \kappa_\star$ can be computed, through lengthy but simple computations, following the proof in [110]. Finally, we remark that similar results are also contained in [24] and [139]. We finally give the details of the lower bound of (6.2.2).

Proposition 6.2.6. *Assume that $m \in (m_1, 1)$. There exist nonnegative constants b such that, if $u(t,x)$ is a solution of (6.1.6) with non-negative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and let $R_\star > 0$ bu such that satisfying $\int_{|x|\le R_\star} u_0\,dx = \mathcal{M}/2$, then*

$$u(t,x) \ge B(t-\frac{t_1}{2}-\frac{1}{\alpha}, x; b\,\mathcal{M}) = \frac{\left(t-\frac{t_1}{2}\right)^{\frac{1}{1-m}}}{\left[\frac{\mathcal{M}^{\frac{2(1-m)}{\alpha}}}{(b\mathcal{M})^{\frac{2(1-m)}{\alpha}}}\frac{\left(t-\frac{t_1}{2}\right)^{\frac{2}{\alpha}}}{\mu^{d(1-m)}\alpha^{d(m-1)}} + \frac{1-m}{2m}\frac{|x|^2}{\alpha}\right]^{\frac{1}{1-m}}} \quad \forall\, t \ge t_1\,, \tag{6.2.9}$$

where t_1 is as in (6.2.7) and b is

$$b \le \mathcal{M}\,\kappa_\star^{\frac{1}{1-m}}\left(\frac{\alpha}{\mu}\right)^{\frac{d\alpha}{2}}\min\left\{\kappa^{\frac{\alpha}{2}}2^{\frac{2}{d}}\,,\kappa\,(2md)^{\frac{\alpha-2}{2(1-m)}}\,\alpha^{-\frac{d\alpha}{2}}\alpha^{\frac{\alpha(1-m)-2}{2(1-m)}}\,,(2md)^{\frac{\alpha(1-m)}{2}}\,2^{-\frac{1}{1-m}}\right\}. \tag{6.2.10}$$

6.2.3 Relative uniform estimates on the tail

From now on we shall use the self-similar variables. Assume that $m \in (m_1, 1)$, as a consequence of Proposition 6.2.2 and Proposition 6.2.6 we obtain that if $u(t,x)$ is a solution of (6.1.6) with non-negative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying (H_A) and let $R_\star > 0$ be such that $\int_{|x|\le R_\star} u_0\,dx = \mathcal{M}/2$, then

$$\text{for any}\quad t\,\ge\,\max\{t_0, t_1\}\,,$$

where t_0 is as in Proposition 6.2.2 and t_1 is as in Proposition 6.2.6, we have that

$$B(t-\frac{t_1}{2}-\frac{1}{\alpha}, x; b\,\mathcal{M})\;\le\;u(t,x)\;\le B(t+\bar{t}-\frac{1}{\alpha}, x; \overline{M}). \tag{6.2.11}$$

In self-similar variable inequality (6.2.11) reads as

$$x(\tau)^d\mathcal{B}\left(y\,x(\tau); b\,\mathcal{M}\right) \le v(\tau, y) \le z(\tau)^d\mathcal{B}\left(y\,z(\tau); \overline{M}\right)\,, \tag{6.2.12}$$

for any $\tau \ge (1/2)\log R(\max\{t_0, t_1\})$, where

$$x(\tau) = \frac{2^{\frac{1}{\alpha}}\,e^{2\tau}}{(2e^{2\tau\,\alpha}-2-\alpha\,t_1)^{\frac{1}{\alpha}}} \quad\text{and}\quad z(\tau) = \frac{e^{2\tau}}{\left(e^{2\tau\,\alpha}-1+\alpha\,\bar{t}\right)^{\frac{1}{\alpha}}}. \tag{6.2.13}$$

Theorem 6.2.7. *Assume that $m \in (m_1, 1)$ and $0 < \varepsilon < 1/2$. There exist non-negative constants $\overline{R}_\varepsilon, \overline{\tau}_\varepsilon$ such that, if v is a solution of* (6.1.14) *with a non-negative initial datum u_0 satisfying* (H_A), *then for any $\tau \geq \overline{\tau}_\varepsilon$*

$$(1-\varepsilon)\; B(y; b\,\mathcal{M}) \leq v(\tau, y) \leq (1+\varepsilon)\; B(y; \overline{M}) \tag{6.2.14}$$

and for any $\tau \geq \overline{\tau}_\varepsilon$ and any $|y| \geq R_\varepsilon$

$$1-\varepsilon \leq \frac{v(s,y)}{\mathcal{B}(y)} \leq 1+\varepsilon\,. \tag{6.2.15}$$

The constants $\overline{R}_\varepsilon, \overline{\tau}_\varepsilon$ have an explicit expression given at the end of the proof.

Proof Let us first prove inequality (6.2.14). Our starting point is inequality (6.2.11). A simple computation shows that the term $\mathcal{B}\left(y\,z(\tau); \overline{M}\right)$ can be estimated as follows

$$(z(\tau)\vee 1)^{\frac{-2}{1-m}}\;\mathcal{B}\left(y;\overline{M}\right) \;\leq\; \mathcal{B}\left(y\,z(\tau);\overline{M}\right) \;\leq\; (z(\tau)\wedge 1)^{\frac{-2}{1-m}}\;\mathcal{B}\left(y;\overline{M}\right)\,.$$

Combining the above inequality with (6.2.12) we can deduce that

$$(x(\tau)\wedge 1)^{d}\;(x(\tau)\vee 1)^{\frac{-2}{1-m}}\;B(y; b\,\mathcal{M}) \leq v(\tau, y) \;\leq (z(\tau)\vee 1)^{d}\;(z(\tau)\wedge 1)^{\frac{-2}{1-m}}\;\mathcal{B}\left(y;\overline{M}\right)\,,$$

to prove (6.2.14) it only remains to estimate $x(\tau)$ and $z(\tau)$. From (6.2.13) we deduce that $x(\tau)\geq 1$ for any $\tau \geq (1/2)\log R(\max\{t_0, t_1\})$, however, to deduce that $z(\tau)\leq 1$ the additional hypothesis $\alpha\,\overline{t} \geq 1$ is needed: this happens to hold in our case since in Proposition 6.2.2 we assume that $t_0 \geq \alpha^{-1}\left(\frac{1-m}{2^8\,m}\,(2\,\kappa_2)^{1-m} - \alpha\right)^{-1}$ which is an equivalent statement, see Remark 6.2.3. Since $x(\tau)$ is increasing and $z(\tau)$ is decreasing, assuming that $\tau \geq \overline{\tau}_\varepsilon$, where

$$\overline{\tau}_\varepsilon = \frac{1}{2\,\alpha}\;\log\max\left\{\frac{2+\alpha\,t_1}{2\left[1-\left(\frac{3-\varepsilon}{3}\right)^{\frac{\alpha(1-m)}{2}}\right]}\,,\;\frac{\alpha\,\overline{t}-1}{\left[\left(\frac{3+\varepsilon}{3}\right)^{\frac{\alpha(1-m)}{2}}-1\right]}\,, 1+\alpha\,\max\left(t_1, t_0\right)\right\}, \tag{6.2.16}$$

is enough to prove that $x(\tau)^{\frac{-2}{1-m}} \leq (1-(\varepsilon/3))$ and that $z(\tau)^{\frac{-2}{1-m}} \geq (1-(\varepsilon/3))$, concluding therefore that inequality (6.2.14) holds. As a consequence, to deduce (6.2.15) we only need to find $\overline{R}_\varepsilon$ such that for any $|y| \geq \overline{R}_\varepsilon$ we have that

$$1-\frac{\varepsilon}{3} \leq \frac{B(y; b\,\mathcal{M})}{\mathcal{B}(y)} \quad\text{and}\quad \frac{\mathcal{B}\left(y;\overline{M}\right)}{\mathcal{B}(y)} \leq 1+\frac{\varepsilon}{3}\,.$$

A simple though lengthy computation shows that

$$\overline{R}_\varepsilon^2 = \max\left\{\frac{\left|1-\left(\frac{3+\varepsilon}{3}\right)^{1-m}\left(\frac{\mathcal{M}}{\overline{M}}\right)^{\frac{2(1-m)}{\alpha}}\right|}{\left[\left(\frac{3+\varepsilon}{3}\right)^{1-m}-1\right]}\,,\;\frac{\left|\left(\frac{3-\varepsilon}{3\,b^{\frac{2}{\alpha}}}\right)^{1-m}-1\right|}{\left[1-\left(\frac{3-\varepsilon}{3}\right)^{1-m}\right]}\right\}, \tag{6.2.17}$$

where $\overline{M}$ is given at the end of the proof of Proposition 6.2.2 and b in (6.2.10). The proof is then concluded. $\square$

Remark 6.2.8. Both inequalities (6.2.11) and (6.2.14) represent a remarkable fact: after some time a solution to (6.1.6) (respectively a solution to (6.1.14)) which satisfy (H_A) is trapped between two Barenblatt profiles. This gives us an insight on the *intermediate asymptotic* of solutions to (6.1.6) (respectively (6.1.14)). First it reveals that (H_A) is sufficient to control the tail of the solution. Indeed, (H_A) is also a necessary condition for (6.2.11) to hold, as proved in chapter 4. Second it suggests that a result such as Theorem 6.2.1 is possible. However, inequality (6.2.11) (inequality (6.2.14) respectively) is not enough to deduce the precise quantitative behaviour in relative uniform norm as in Theorem 6.2.1. The main obstruction is the same in both original and self-similar variables, however it is easier to understand in self-similar variables. The Barenblatt profiles $B(y; b\,\mathcal{M}), \mathcal{B}\left(y; \overline{M}\right)$ which control the pointwise behaviour of the solution $v(\tau, y)$ have different masses with respect to $\mathcal{M}$, i.e. the mass of the solution $v(\tau, y)$. This allows to give a precise control of the tail of $v(\tau, y)$ since $B(y; b\,\mathcal{M}) \sim \mathcal{B}\left(y; \overline{M}\right)$ as $|y| \to \infty$, but it is not enough to control the quotient $v(\tau, y)/\mathcal{B}(y)$ close to the origin. Indeed the supremum of $\mathcal{B}\left(y; \overline{M}\right)/\mathcal{B}(y)$ and the infimum of $B(y; b\,\mathcal{M})/\mathcal{B}(y)$ are both achieved at the origin and are not equal to 1, which makes impossible to deduce Theorem 6.2.1 from inequality (6.2.14). Therefore more work is needed.

6.2.4 Hölder continuity and relative uniform estimates in a ball

We prove a precise quantitative control of $v(\tau, y)/\mathcal{B}(y)$ on a ball centered in the origin. This result is complementary to inequality (6.2.15) of Theorem 6.2.7 and is the missing part in the proof of Theorem 6.2.1, see Remark 6.2.8.

Theorem 6.2.9. *Assume that $m \in (m_1, 1)$ and let $r > 0$. There exist positive constants C, ϑ such that, if v is a solution of* (6.1.14) *with a non-negative initial datum u_0 satisfying* (H_A)*, then*

$$\sup_{x \in B_r(0)} \left|\frac{v(\tau, x)}{\mathcal{B}(x)} - 1\right| \le C\,\left(1 + r^2\right)^{\frac{1}{1-m}}\,\left[\mathcal{F}[u_0]\,e^{-4\tau}\right]^{\frac{1-\vartheta}{2}} \quad \forall\,\tau \ge \overline{\tau}_{\frac{1}{4}}\,, \tag{6.2.18}$$

where C, ϑ are given at the end of the proof and $\overline{\tau}_{\frac{1}{4}}$ is as in Theorem 6.2.7.

The proof of the above Theorem is based on a explicit estimate of the $C^\nu(\mathbb{R}^d)$ norm of the solution $v(\tau, y)$, where the $C^\nu(\mathbb{R}^d)$-seminorm is defined as follows:

$$\lfloor v \rfloor_{C^\nu(\mathbb{R}^d)} := \sup_{\substack{x, y \in \mathbb{R}^d \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\nu}\,. \tag{6.2.19}$$

To obtain such estimates we shall use parabolic regularity theory, but we have to be careful to choose a strategy which is both quantitative at any step (no compactness nor contradiction arguments) and which can be adapted to our nonlinear-singular case. The main idea is to treat the nonlinear equation (6.1.6) as a linear one with measurable coefficients, which need to be uniformly parabolic, see Appendix 6.6. Unfortunately, the coefficients depend on the nonlinearity and are a priori singular/degenerate. To overcome this difficulty and obtain uniform C^ν estimates, the quantitative upper and lower bounds obtained in the previous sections will play an essential role, more specifically inequality (6.2.14). Adapting the strategy of the pioneering paper [74] to our setting, as in [139], we are able to quantify all the constants and obtain a quantitative uniform C^ν continuity on the whole $\mathbb{R}^d$. Therefore, we obtain a quantitative version of the analogous estimates appearing in [128, 19].

Lemma 3. *Under the same assumption of Theorem 6.2.9 there exists a nonnegative number* $\nu \in (0,1)$ *such that*

$$\lfloor v(\tau,\cdot)-\mathcal{B}\rfloor_{C^\nu(\mathbb{R}^d)} \le 2\,\frac{2^\nu}{2^\nu-1}\,\frac{5}{4}\left(\frac{16\,R(3)}{R(1)}\right)\left(\frac{\overline{M}}{\mathcal{M}}\right)^{\frac{2}{\alpha}}\left(\frac{\mu}{\alpha}\right)^d \quad \text{for any } \tau \ge \overline{\tau}_{\frac14}\,. \tag{6.2.20}$$

The constant ν *depends only on* d, m *and is given at the end of the proof,* $\overline{\tau}_{\frac14}$ *is as in Theorem 6.2.7 and* $\overline{M}$ *is as in Proposition 6.2.2.*

Proof We split the proof in several steps.
Step 1. In this step we want to prove the following estimates

$$\begin{aligned}
\lfloor v(\tau,\cdot)\rfloor_{C^\nu(B_1(0))} &\le \frac{5}{4}\left(\frac{16\,R(3)}{R(1)}\right)^\nu \|\mathcal{B}\|_{L^\infty(B_8(0))}\,,\\
\lfloor v(\tau,\cdot)\rfloor_{C^\nu\left(B_1(0)\setminus B_{1/2}(0)\right)} &\le \frac{5}{4}\left(\frac{16\,R(3)}{R(1)}\right)^\nu \|\mathcal{B}\|_{L^\infty(B_8(0)\setminus B_{1/4}(0)}\,,
\end{aligned} \tag{6.2.21}$$

for any $\tau \ge \overline{\tau}_{\frac14}$ and for some positive ν. We proceed as follows. Regularity estimates as inequality (6.6.1) in Appendix 6.6 are technically easier to obtain if the equation is in the form (6.1.6), so we shall use the change of variable 6.1.14 to obtain the desired result. We shall exploit the idea that a solution to (6.1.6) satisfies a linear parabolic equation whose coefficient is $m\,u^{m-1}$, therefore is key to obtain estimates from above and below for such quantity. Let us define the parabolic cylinders

$$Q_1 \quad := \quad \left[\frac12\log R_*(1), \frac12\log R_*(2)\right] \;\times\; B_1(0)\,, \qquad Q_2 \quad := \quad \left[0, \frac12\log R_*(3)\right] \;\times\; B_8(0)\,,$$

and

$$A_1 := \left[\frac12\log R_*(1), \frac12\log R_*(2)\right] \times B_1(0)\setminus B_{1/2}(0)\,, \qquad A_2 := \left[0, \frac12\log R_*(3)\right] \times B_8(0)\setminus B_{1/4}(0)\,,$$

it is clear that $Q_1 \subset Q_2$ and $A_1 \subset A_2$. Without loss of generality we can use the semigroup property and work with the function $\tilde v(\tau,y) = v(\tau+\overline{\tau}_{\frac14}, y)$. As a consequence of (6.2.14) (with $\varepsilon = 1/4$) we obtain the following inequality for $\tilde v$ which holds on both Q_2 and A_2

$$\begin{aligned}
m\left(\frac45\right)^{1-m}\min\left\{\frac{1}{16}, \left(\frac{\mathcal{M}}{\overline{M}}\right)^{2\alpha(1-m)}\right\} &\le m\tilde v^{m-1}(\tau,y)\\
&\le 2m\left(\frac43\right)^{1-m}\max\left\{\frac{\alpha^{d(1-m)}}{\mu^{d(1-m)}\,b^{\frac{2(1-m)}{\alpha}}}, \frac{64(1-m)}{2\alpha m}\right\}.
\end{aligned} \tag{6.2.22}$$

Let us first procced with the couple of cylinders Q_1, Q_2, the procedure will be analogous for A_1 and A_2. We perform the change of variables (6.1.13) and pass from $\tilde v$ which satisfies equation (6.1.14) on Q_2 to $\tilde u$ which satisfies equation (6.1.6) on $\tilde Q_2$ where $\tilde Q_2$ is the image of Q_2 in the change of variables. We observe then that inequality (6.2.22) scales accordingly to the change of variables. We are now in the position to apply inequality (6.6.3) of Proposition 6.6.1 to $\tilde u(t,x)$ assuming that $a(t,x) = m\tilde u(t,x)^{m-1}$. The values for λ_0 and λ_1 are

$$\lambda_0 = \frac{m}{\mu^{d(1-m)}}\left(\frac45\right)^{1-m}\min\left\{\frac{1}{16}, \left(\frac{\mathcal{M}}{\overline{M}}\right)^{2\alpha(1-m)}\right\} \quad \text{and}$$

$$\lambda_1 = 2\frac{mR(3)^{d(1-m)}}{\mu^{d(1-m)}}\left(\frac43\right)^{1-m}\max\left\{\frac{\alpha^{d(1-m)}}{\mu^{d(1-m)}\,b^{\frac{2(1-m)}{\alpha}}}, \frac{64(1-m)}{2\alpha m}\right\}, \tag{6.2.23}$$

and $d_p\left(\tilde{Q}_1, \tilde{Q}_2\right) = 7R(1)/\mu$, we conclude then that for any $(t,x),(s,z) \in \tilde{Q}_1$

$$|\tilde{u}(t,x) - \tilde{u}(s,z)| \le \left(\frac{8\,\mu}{7\,R(1)}\right)^{\nu} \|\tilde{u}\|_{L^\infty(\tilde{Q}_2)} \left(|x-z| + |t-s|^{1/2}\right)^{\nu}, \tag{6.2.24}$$

the value of ν is given by

$$\nu := \log_8 \frac{\kappa_\ell^{\lambda_0^{-1}+\lambda_1}}{\kappa_\ell^{\lambda_0^{-1}+\lambda_1} - 1}, \tag{6.2.25}$$

where $\kappa_\ell \ge 2$ and depends only on the dimension d. We claim that inequality (6.2.24) implies (6.2.21). Indeed, we obtain (6.2.24) by considering 6.2.24 for points that have the same time coordinate and then rescale back to the function $v(\tau,y)$. The proof of the first inequality in (6.2.24) is then complete. To obtain the second one we apply the same procedure to the couple of parabolic cylinders A_1, A_2. The only difference is the value of the parabolic distance among $d_p\left(\tilde{A}_1, \tilde{A}_2\right) = R(1)/(2\mu)$. The proof of this step is concluded.

Step 2. In this step we shall extend inequality 6.2.21 to sets of the form $B_\lambda(0) \setminus B_{\lambda/2}(0)$ for every $\lambda \ge 1$. We consider the auxiliary function

$$v_\lambda(\tau,y) = \lambda^{\frac{2}{1-m}}\, v(\tau,\lambda y)\,,$$

which is a solution to (6.1.14). As a consequence of (6.2.14), $v_\lambda(\tau,y)$ satisfies inequality (6.2.22) on the parabolic cylinder Q_2. Therefore we may apply the procedure already explained in *Step 1* and get inequality (6.2.21) as a result. Once we scale back to $v(\tau,y)$ we obtain the following

$$\lfloor v(\tau,\cdot)\rfloor_{C^\nu\left(B_\lambda(0)\setminus B_{\lambda/2}(0)\right)} \le \frac{5}{4}\left(\frac{16\,R(3)}{R(1)}\right)^{\nu} \|\mathcal{B}\|_{L^\infty(B_\lambda(0))\setminus B_{\lambda/4}(0)}\,\lambda^{-\nu}\,.$$

Step 3. In this last step we join the inequalities obtained in *Step 1* and *Step 2* to get the desired result. As final remark we observe that since $\mathcal{B}$ satisfies inequality (6.2.14) all the estimates developed in *Step 1* and *Step 2* applies to $\mathcal{B}$. We consider the following chain of inequalities, which hold for any $\tau \ge \overline{\tau}_{\frac{1}{4}}$

$$\begin{aligned}
\lfloor v(\tau,\cdot) - \mathcal{B}\rfloor_{C^\nu\left(\mathbb{R}^d\right)} &\le \lfloor v(\tau,\cdot)\rfloor_{C^\nu\left(\mathbb{R}^d\right)} + \lfloor \mathcal{B}\rfloor_{C^\nu\left(\mathbb{R}^d\right)}\\
&\le 2\left(\sum_{j=0}^{\infty} \lfloor v(\tau,\cdot)\rfloor_{C^\nu\left(B_{2^{j+1}}(0)\setminus B_{2^j}(0)\right)} + \lfloor v(\tau,\cdot)\rfloor_{C^\nu(B_1(0))}\right)\\
&\le 2\left(\frac{5}{4}\left(\frac{16\,R(3)}{R(1)}\right)^{\nu} \sum_{j=0}^{\infty} \|\mathcal{B}\|_{L^\infty(B_{2^{j+1}})(0)\setminus B_{2^j}(0)}\, 2^{-j\,\nu} + \frac{5}{4}\left(\frac{16\,R(3)}{R(1)}\right)^{\nu} \|\mathcal{B}\|_{L^\infty(B_8(0))}\right)\\
&\le 2\,\frac{2^\nu}{2^\nu - 1}\,\frac{5}{4}\left(\frac{16\,R(3)}{R(1)}\right)\left(\frac{\overline{M}}{\mathcal{M}}\right)^{\frac{2}{\alpha}}\left(\frac{\mu}{\alpha}\right)^{d}
\end{aligned}$$

where in the very last line we have used the fact that $\|B(y;\overline{M})\|_{L^\infty(\mathbb{R}^d)} = \left(\frac{\overline{M}}{\mathcal{M}}\right)^{\frac{2}{\alpha}}\left(\frac{\mu}{\alpha}\right)^{d}$. □

Finally we are in the position to prove Theorem 6.2.9.

Proof of Theorem 6.2.9 The following computation holds for any $r > 0$ and for any $\tau > 0$

$$\sup_{y\in B_r(0)} \left|\frac{v(\tau,y)}{\mathcal{B}(y)} - 1\right| \le \|v(\tau,\cdot) - \mathcal{B}\|_{L^\infty(\mathbb{R}^d)} \left(1+r^2\right)^{\frac{1}{1-m}}$$
$$\le C_{d,2,\nu} \left(1+r^2\right)^{\frac{1}{1-m}} \|v(\tau,\cdot)-\mathcal{B}\|^{\vartheta}_{C^\nu(\mathbb{R}^d)} \|v(\tau,\cdot)-\mathcal{B}\|^{1-\vartheta}_{L^2(\mathbb{R}^d)},$$

where we have used the explicit expression of $\mathcal{B}$ and the interpolation inequality (6.4.1) with $p = 2$, ν being as in (6.2.25) and

$$\vartheta = \frac{d}{d+2\nu}.$$

We now estimate the L^2 norm of the difference $v(\tau, y) - \mathcal{B}$, since $\mathcal{B}^{m-2} \ge 1$ we have that

$$\|v(\tau,\cdot) - \mathcal{B}\|_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |v(\tau,y) - \mathcal{B}(y)|^2\,dy \le \int_{\mathbb{R}^d} \left|\frac{v(\tau,y)}{\mathcal{B}(y)} - 1\right|^2 \mathcal{B}^m\,dy.$$

For $\tau > \overline{\tau}_{\frac14}$ inequality 6.2.14 holds with $\varepsilon = 1/4$, therefore we are in the position of using inequality (6.5.1) of Lemma 7 with

$$(1+\varepsilon) = \frac54 \sup_{y\in\mathbb{R}^d} \frac{B(y;\overline{M})}{\mathcal{B}(y)} = \frac54 \left(\frac{\overline{M}}{\mathcal{M}}\right)^{\frac{2}{\alpha}} \left(\frac{\mu}{\alpha}\right)^d$$

This yields to inequality (6.2.18) with constant

$$C = C_{d,2,\nu}\, 2\, \frac{2^\nu}{2^\nu - 1}\, \frac54 \left(\frac{16\,R(3)}{R(1)}\right) \left(\frac{\overline{M}}{\mathcal{M}}\right)^{\frac{2}{\alpha}} \left(\frac{\mu}{\alpha}\right)^d \left(\frac{2}{m}\,b^{2-m}\right)^{\frac{1-\vartheta}{2}}. \tag{6.2.26}$$

The proof is concluded. $\square$

6.2.5 Proof of Theorem 6.2.1

Proof of Theorem 6.2.1 The main idea is to combine inequality (6.2.15), which provides a control of the tail of v, with inequality (6.2.18) which gives us an explicit control of the asymptotic of $v(\tau, y)$ on a small ball centered in the origin. Let $0 < \varepsilon < 1/2$. For any $\tau \ge \overline{\tau}_\varepsilon$ we have that

$$\left|\frac{v(\tau,y)}{\mathcal{B}(y)} - 1\right| < \varepsilon \quad \text{for any} |y| \ge \overline{R}_\varepsilon,$$

where $\overline{\tau}_\varepsilon$ and $\overline{R}_\varepsilon$ are given in (6.2.16) and (6.2.17) respectively. It remains to prove the previous estimate in $|y| \le \overline{R}_\varepsilon$. Let $\hat{\tau}$ be the smallest time such that

$$C\left(1+\overline{R}_\varepsilon^2\right)^{\frac{1}{1-m}} \left[\mathcal{F}[u_0]\,e^{-4\hat{\tau}}\right]^{\frac{1-\vartheta}{2}} < \varepsilon, \tag{6.2.27}$$

where C is given in (6.2.26). Then using inequality (6.2.18) we have that, for any $\tau \ge \max\left\{\overline{\tau}_{\frac14}, \hat{\tau}\right\}$

$$\left|\frac{v(\tau,y)}{\mathcal{B}(y)} - 1\right| < \varepsilon \quad \text{for any} \quad |y| \le \overline{R}_\varepsilon.$$

Therefore both estimates hold when

$$\tau \ge \max\left\{\hat{\tau}, \overline{\tau}_\varepsilon\right\}.$$

Rescaling back to the original variables we can find the value of $t_\star(A)$. The proof is concluded. $\square$

6.3 Entropy estimates in self-similar variables and consequences

6.3.1 Improved entropy-entropy production inequality along the flow

In this section, we adapt the method of [128] to the solution of (6.1.14) to obtain an improved entropy-entropy production inequality for solutions which satisfy the following

$$1-\varepsilon \le \frac{v(\tau,y)}{\mathcal{B}(y)} \le 1+\varepsilon \quad \forall\, \tau \ge \tau_\star\,, \quad y \in \mathbb{R}^d\,. \tag{6.3.1}$$

We will make extensive use of the function

$$w(\tau,y) = \frac{v(\tau,y)}{\mathcal{B}(y)}\,. \tag{6.3.2}$$

In the variable w the free energy is given by

$$\mathcal{F}[w(\tau)] = \frac{m}{m-1}\int_{\mathbb{R}^d}\left[\frac{w^m-1}{m}-(w-1)\right]\mathcal{B}^m dy\,,$$

while the entropy production (or fisher information)

$$\mathcal{I}[v] = \frac{m}{1-m}\int_{\mathbb{R}^d} w\,\mathcal{B}\Big|\nabla\left[\left(w^{m-1}-1\right)\mathcal{B}^{m-1}\right]\Big|^2 dy\,.$$

Finally, let us define the linearized *Fisher Information* $I[g]$, where $g := (w-1)\mathcal{B}^{m-1}$,

$$I[g] = \int_{\mathbb{R}^d}|\nabla g|^2\mathcal{B}dy = \int_{\mathbb{R}^d}\Big|\nabla\left[(w-1)\mathcal{B}^{m-1}\right]\Big|^2\mathcal{B}dy \tag{6.3.3}$$

The next Lemma compares the linearized Fisher information with the nonlinear one.

Lemma 4. *Assume that* $m \in (m_1,1)$, $0<\varepsilon<1$ *and assume that* (6.3.1) *holds. Then we have that*

$$I[g] \le \frac{s_1(\varepsilon)}{(1-m)m}\mathcal{I}[w] + 2d\,s_2(\varepsilon)\int_{\mathbb{R}^d}|w-1|^2\,\mathcal{B}^m dy\,, \tag{6.3.4}$$

with

$$s_1(\varepsilon) = \frac{(1+\varepsilon)^{2(2-m)}}{(1-\varepsilon)} \qquad \textit{and} \qquad s_2(\varepsilon) = \left(\frac{(1+\varepsilon)^{2(2-m)}}{(1-\varepsilon)^{2(2-m)}}-1\right)\,.$$

Proof Let us first give an idea of the proof. We need to compare the linearized Fisher information with the nonlinear one. In order to do so, we introduce a third quantity, namely

$$\mathfrak{I}[w] = \int_{\mathbb{R}^d}\Big|\nabla\left[\left(w^{m-1}-1\right)\mathcal{B}^{m-1}\right]\Big|^2\mathcal{B}dy$$

which behaves very much like the nonlinear Fisher information. So we will first compare the $I[g]$ with $\mathfrak{I}[w]$ and then conclude by observing that (6.3.1) implies $\mathfrak{I}[w] \le (1-\varepsilon)^{-1}\,((1-m)/m)\,\mathcal{I}[w]$.

Recall that $g=(w-1)\mathcal{B}^{m-1}$, a simple computation then shows that

$$I[g] = \int_{\mathbb{R}^d}|\nabla w|^2\mathcal{B}^{2m-1}dy + \frac{4}{1-m}\int_{\mathbb{R}^d}|y|^2\,|w-1|^2\mathcal{B}dy - 2d\int_{\mathbb{R}^d}|w-1|^2\mathcal{B}^m dy\,.$$

While $\mathfrak{I}[w]$ can be written as

$$\begin{aligned}\mathfrak{I}[w] = &\int_{\mathbb{R}^d} \left|\nabla\left(w^{m-1}-1\right)\right|^2 \mathcal{B}^{2m-1} dy + \\ &\frac{4}{1-m}\int_{\mathbb{R}^d} |y|^2\,|w^{m-1}-1|^2\,\mathcal{B} dy - 2d\int_{\mathbb{R}^d} |w^{m-1}-1|^2 \mathcal{B}^m dy\,.\end{aligned} \tag{6.3.5}$$

We notice that, since $w \in (1-\varepsilon, 1+\varepsilon)$ we have that

$$\alpha_0\,\frac{|w^{m-1}-1|^2}{(1-m)^2} \le |w-1|^2 \le \alpha_1\,\frac{|w^{m-1}-1|^2}{(1-m)^2}\,, \tag{6.3.6}$$

where

$$\alpha_0 = (1-\varepsilon)^{2(2-m)} \qquad \text{and} \qquad \alpha_1 = (1+\varepsilon)^{2(2-m)}\,.$$

As well we have that $|\nabla\left(w^{m-1}-1\right)|^2 = (1-m)^2\,w^{2(m-2)}\,|\nabla w|^2$ and as a consequence

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla w|^2 \mathcal{B}^{2m-1} dy &= \frac{(1-m)^2}{(1-m)^2}\int_{\mathbb{R}^d} w^{2(m-2)}\,w^{2(2-m)}\,|\nabla w|^2\,\mathcal{B}^{2m-1} dy \\ &\le \frac{\alpha_1}{(1-m)^2}\int_{\mathbb{R}^d} \left|\nabla\left(w^{m-1}-1\right)\right|^2 \mathcal{B}^{2m-1} dy\,.\end{aligned} \tag{6.3.7}$$

As a consequence of (6.3.6) we also have the following

$$\int_{\mathbb{R}^d} |y|^2\,|w-1|^2 \mathcal{B} dy \le \frac{\alpha_1}{(1-m)^2}\int_{\mathbb{R}^d} |y|^2\,|w^{m-1}-1|^2 \mathcal{B} dy\,.$$

Collecting the above computations together we obtain

$$\begin{aligned}I[g] \le &\frac{\alpha_1}{(1-m)^2}\int_{\mathbb{R}^d} |\nabla\left(w^{m-1}-1\right)|^2\,\mathcal{B}^{2m-1}\,dy \\ &+ \frac{4\,\alpha_1}{(1-m)^3}\int_{\mathbb{R}^d} |y|^2\,|w^{m-1}-1|^2 \mathcal{B} dy - 2d\int_{\mathbb{R}^d} |w-1|^2 \mathcal{B}^m dy\,.\end{aligned} \tag{6.3.8}$$

As a consequence of (6.3.5) we obtain the following expression

$$\begin{aligned}I[g] \le &\frac{\alpha_1}{(1-m)^2}\left[\mathfrak{I}[w] - \frac{4}{1-m}\int_{\mathbb{R}^d} |y|^2\,|w^{m-1}-1|^2 \mathcal{B} dy + 2d\int_{\mathbb{R}^d} |w^{m-1}-1|^2 \mathcal{B}^m dy\right] \\ &+ \frac{4\,\alpha_1}{(1-m)^3}\int_{\mathbb{R}^d} |y|^2\,|w^{m-1}-1|^2 \mathcal{B} dy - 2d\int_{\mathbb{R}^d} |w-1|^2 \mathcal{B}^m dy\,.\end{aligned}$$

Simplifying and collecting terms in the above inequality we get

$$I[g] \le \frac{\alpha_1}{(1-m)^2}\,\mathfrak{I}[w] + 2d\int_{\mathbb{R}^d} |w^{m-1}-1|^2 \mathcal{B}^m dy - 2d\int_{\mathbb{R}^d} |w-1|^2 \mathcal{B}^m dy\,.$$

As a consequence of the fact that $w \in (1-\varepsilon, 1+\varepsilon)$ we obtain

$$I[g] \le \frac{\alpha_1}{(1-\varepsilon)(1-m)m}\,\mathcal{I}[w] + 2d\left(\frac{\alpha_1}{\alpha_0} - 1\right)\int_{\mathbb{R}^d} |w-1|^2 \mathcal{B}^m dy\,. \tag{6.3.9}$$

The proof is then complete. □

As a consequence of Lemma 4 we are able to prove an improved entropy-entropy production inequality which holds along the flow (6.1.14) for large times, namely when (6.3.1) holds.

Proposition 6.3.1. *Assume that $m \in (m_1, 1)$ and $\varepsilon \in (0,1)$. If v is a solution of (6.1.14) with initial datum v_0 such that $\int_{\mathbb{R}^d} (1,x)\, u_0\, dx = \int_{\mathbb{R}^d} (1,x)\, \mathcal{B}\, dx$, $\mathcal{F}[v_0] < \infty$ and $\mathcal{I}[v_0] < \infty$ satisfying (6.3.1), then we have the estimate*

$$\mathcal{I}[v(\tau)] \geq \mathcal{C}_\infty(\varepsilon)\, \mathcal{F}[v(\tau)] \quad \forall\, \tau \geq \tau_\star \tag{6.3.10}$$

with

$$\mathcal{C}_\infty(\varepsilon) = 4 \left[2 \left(\frac{1-\varepsilon}{1+\varepsilon} \right)^{2(2-m)} - d(1-m) \right].$$

Remark 6.3.2. *As it is clear from the expression of $C_\infty(\varepsilon)$ we have that for $\varepsilon \to 0$ that*

$$C_\infty(\varepsilon) \to 4\alpha = 4(2 - d(1-m)) > 4 \quad \textit{when} \quad m \in (m_1, 1)\,.$$

Since $C_\infty(\varepsilon)$ is decreasing for $\varepsilon \in (0,1)$ there exists a unique $\varepsilon_{m,d} \in (0,1)$ such that

$$C_\infty(\varepsilon_{m,d}) = 4\,.$$

Proof. The proof is a consequence of the following Hardy-Poincaré inequality proven in [129, Lemma 1], see also [153, Proposition 1]. □

Lemma 5. *Let $d \geq 3$ and $m \in (m_1, 1)$. Assume that $f \in \mathrm{L}^1(\mathbb{R}^d, \mathcal{B}^{2-m} dx)$ and $\nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B} dx)$. Then the following inequality holds*

$$2 \left(\frac{2 - d(1-m)}{1-m} \right) \int_{\mathbb{R}^d} |g|^2\, \mathcal{B}^{2-m}\, dx \leq \int_{\mathbb{R}^d} |\nabla g|^2\, \mathcal{B}\, dx \tag{6.3.11}$$

under the conditions

$$\int_{\mathbb{R}^d} g\, \mathcal{B}^{2-m}\, dx = 0 \qquad \textit{and} \qquad \int_{\mathbb{R}^d} x\, g\, \mathcal{B}^{2-m}\, dx = 0\,.$$

We notice that in the case of $g = (w-1)\, \mathcal{B}^{m-1}$ we have that

$$\int_{\mathbb{R}^d} |\nabla g|^2\, \mathcal{B}\, dx = I[g] \qquad \text{and} \qquad \int_{\mathbb{R}^d} |g|^2\, \mathcal{B}^{2-m}\, dx = \int_{\mathbb{R}^d} |w-1|^2\, \mathcal{B}^m\, dx\,.$$

The conditions $\int_{\mathbb{R}^d} g\, \mathcal{B}^{2-m}\, dx = 0$ and $\int_{\mathbb{R}^d} x\, g\, \mathcal{B}^{2-m}\, dx = 0$ are satisfied since the former is the conservation of mass for $v(\tau)$ while the latter is the conservation of the center on mass once it is in $0 \in \mathbb{R}^d$.

Combining inequality (6.3.11) with inequality (6.3.4) of Lemma 4 we obtain

$$\left[2 \frac{2 - d(1-m)}{1-m} - 2d\, s_2(\varepsilon) \right] \int_{\mathbb{R}^d} |w-1|^2\, \mathcal{B}^m\, dx \leq \frac{s_1(\varepsilon)}{(1-m)m}\, \mathcal{I}[w]\,.$$

Then to get inequality (6.3.10) one needs to combine the above inequality with (6.5.1) of Lemma 7. The proof is concluded. □

6.3.2 Improved entropy-entropy production inequality in suitable neighbourhoods of the Barenblatt

In section 6.2 we have proven that $\mathcal{X}_{\mathcal{M}}$ is the *basin of attraction* of $\mathcal{B}$ in relative error, where

$$\mathcal{X}_{\mathcal{M}} := \{v_0 \in \mathrm{L}^1(\mathbb{R}^d) : v_0 \geq 0\,, \|v_0\|_{\mathrm{L}^1(\mathbb{R}^d,dx)} = \mathcal{M}\,, v_0 \quad \text{satisfies } (\mathrm{H}_A)\}\,. \tag{6.3.12}$$

Let $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and assume that it satisfies (H_A), then for any $\varepsilon > 0$ we can define

$$T_\star^\varepsilon(v_0) := \inf\{T \geq 0 : \left\|\frac{v(t)}{\mathcal{B}} - 1\right\|_{L^\infty(\mathbb{R}^d)} < \varepsilon \quad \text{for any} \quad t \geq T\}\,. \tag{6.3.13}$$

Remark 6.3.3. The assumption (H_A) implies that $T_\star^\varepsilon(v_0) < \infty$.

We will now define a covering of the space $\mathcal{X}_{\mathcal{M}}$, for any $\varepsilon > 0$ and for any $N > 0$ let us define

$$\mathcal{V}_\varepsilon^N := \{v_0 \in \mathcal{X}_{\mathcal{M}} : T_\star^\varepsilon(v_0) \leq N\}\,. \tag{6.3.14}$$

Remark 6.3.4. It is not difficult to show that $\{\mathcal{V}_\varepsilon^N\}_{\varepsilon,N}$ is a covering of $\mathcal{X}_{\mathcal{M}}$, indeed

$$\mathcal{X}_{\mathcal{M}} = \bigcup_{\varepsilon \geq 0} \bigcup_{N \geq 0} \mathcal{V}_\varepsilon^N$$

Remark 6.3.5. Notice that the set $\mathcal{V}_\varepsilon^N$ is *stable* under the flow given by (6.1.14). In other words: assume that $v_0 \in \mathcal{V}_\varepsilon^N$ for some $N, \varepsilon > 0$, then, by the semigroup property, we have that

$$v(t) \in \mathcal{V}_\varepsilon^N\,, \quad \text{for any} \quad t > 0\,.$$

Even more, it happens that for any $t \geq$ we have $T_\star^\varepsilon(v(t)) = \max\{T_\star^\varepsilon(v_0) - t\,;\,,0\}$.

In the sets $\mathcal{V}_\varepsilon^N$ an improved entropy-entropy production inequality holds, indeed we have the following proposition.

Proposition 6.3.6. *Assume that $m \in (m_1, 1)$, $0 < \varepsilon < \varepsilon_{m,d}$ and $N \geq 0$. There exists a constant $\mathcal{C}(\varepsilon, N)$ such that for all $v \in \mathcal{V}_\varepsilon^N$ with $\int_{\mathbb{R}^d}(1,x)\,v\,dx = \int_{\mathbb{R}^d}(1,x)\,\mathcal{B}\,dx$, $\mathcal{F}[v] < \infty$ and $\mathcal{I}[v] < \infty$ the following inequality holds*

$$\mathcal{I}[v] - 4\mathcal{F}[v] \geq \mathcal{C}(\varepsilon, N)\,\mathcal{F}[v]\,, \tag{6.3.15}$$

where

$$\mathcal{C}(\varepsilon, N) = 4\frac{e^{-8N}}{1 - e^{-8N}}\min\{1\,, \left(\frac{\mathcal{C}_\infty(\varepsilon)}{4} - 1\right)\left(e^{4N} - 1\right)\}\,. \tag{6.3.16}$$

Proof. Let us consider the Rayleigh quotient $\mathcal{I}[v]/(4\mathcal{F}[v])$. We have the following dichotomy: either

$$\text{(A)} \quad \frac{\mathcal{I}[v]}{4\,\mathcal{F}[v]} \geq 1 + \frac{e^{-8N}}{1 - e^{-8N}}\,,$$

or

$$\text{(B)} \quad \frac{\mathcal{I}[v]}{4\,\mathcal{F}[v]} < 1 + \frac{e^{-8N}}{1 - e^{-8N}}\,.$$

Since in (A) there is nothing to prove let us consider (B) and let us further assume that

$$\left(\frac{\mathcal{C}_\infty(\varepsilon)}{4} - 1\right)\left(e^{4N} - 1\right) \leq 1\,.$$

This further restriction will be overcome at the end of the proof. Let us define $v(\tau)$ as the solution to (6.1.14) with initial data $v(0) = v$. We divide the proof in two steps: in step 1 we show how to estimate the entropy $\mathcal{F}[v(N)]$ by below, as a consequence of this estimate we show in step 2 that (6.3.15) holds.

Step 1. We want to prove that

$$\mathcal{F}[v(N)] \geq e^{-8N}\,\frac{e^{4N}-1}{1-e^{-8N}}\,\mathcal{F}[v(0)]\,. \tag{6.3.17}$$

Integrating the first equality in (6.1.15) and using that $\mathcal{I}[v(\tau)] \leq \mathcal{I}[v(0)]e^{-4\tau}$ (obtained as a consequence of $\frac{d}{d\tau}\mathcal{I}[v(\tau)] \leq -4\mathcal{I}[v(\tau)]$) we obtain

$$\mathcal{F}[v(N)] - \mathcal{F}[v(0)] \geq \mathcal{I}[v(0)]\,\frac{e^{-4N}-1}{4}\,.$$

To obtain (6.3.17) we only need to combine the above inequality with inequality (B). This step is concluded.

Step 2.
Since $v(0) \in \mathcal{V}_\varepsilon^N$ by inequality (6.3.10) we have that at time $\tau = N$

$$\mathcal{I}[v(N)] - 4\mathcal{F}[v(N)] \geq \mathcal{C}(\varepsilon)\mathcal{F}[v(N)]\,.$$

Therefore inequality (6.3.15) is obtained as a consequence of the following chain of inequalities

$$\begin{aligned}\mathcal{I}[v(0)] - 4\mathcal{F}[v(0)] &\geq \mathcal{I}[v(N)] - 4\mathcal{F}[v(N)]\\ &\geq (\mathcal{C}(\varepsilon)-4)\,\mathcal{F}[v(N)]\\ &\geq (\mathcal{C}(\varepsilon)-4)\,e^{-8N}\,\frac{e^{4N}-1}{1-e^{-8N}}\,\mathcal{F}[v(0)]\,,\end{aligned}$$

where we have used the fact that the quantity $\mathcal{I}[v(\tau)] - 4\mathcal{F}[v(\tau)]$ is monotone non increasing under the flow. This concludes the step 2.

It only remains to overcome the restriction $\left(\frac{\mathcal{C}_\infty(\varepsilon)}{4}-1\right)\left(e^{4N}-1\right) \leq 1$ assumed at the beginning. Here we argue by contradiction: assume that (B) holds with $\left(\frac{\mathcal{C}_\infty(\varepsilon)}{4}-1\right)\left(e^{4N}-1\right) > 1$. By repeating the procedure described in step 1 and 2 we deduce that

$$\frac{\mathcal{I}[v(0)]}{4\,\mathcal{F}[v(0)]} \geq 1 + \left(\frac{\mathcal{C}_\infty(\varepsilon)}{4}-1\right)\left(e^{4N}-1\right)\,\frac{e^{-8N}}{1-e^{-8N}} > 1 + \frac{e^{-8N}}{1-e^{-8N}}\,,$$

which contradicts (B). The proof is concluded. □

6.3.3 Proofs of Theorem 6.1.1 and Theorem 6.1.4

As we already explained in Section 6.1.1, Theorem 6.1.1 and Theorem 6.1.4 are two different statements of the same result, therefore it is enough to prove only Theorem 6.1.4.

Proof of Theorem 6.1.4. In view of the result of Proposition 6.3.6 we only need to show that $u_0 \in \mathcal{V}_\varepsilon^N$ for some $\varepsilon > 0$ and $N > 0$. As a consequence of Theorem 6.2.1 we know that for any $\varepsilon > 0$ the time $T_\star^\varepsilon(u_0) < \infty$ and as a consequence of Remark 6.3.5 it happens that

$$u_0 \in \mathcal{V}_\varepsilon^N\,, \qquad \text{for any} \quad N \geq T_\star^\varepsilon(u_0)\,.$$

Our strategy will be to fix $\varepsilon > 0$ to a number which will allow us to use the result of Proposition 6.3.6 and to find $N > 0$ such that $T_\star^\varepsilon(u_0) < N$. Let us fix $\overline{\varepsilon} > 0$ such that

$$\overline{\varepsilon} := \min\left\{\frac{1}{8}, \frac{\varepsilon_{m,d}}{8}\right\},$$

where $\varepsilon_{m,d}$ is as in Remark 6.3.2. It clear from the proof of Theorem 6.2.1 that under our assumptions we have

$$T_\star^\varepsilon(u_0) \le \max\left\{\hat{\tau}, \overline{\tau}_\varepsilon\right\}, \quad \text{for any } 0 < \varepsilon < 1/2\,,$$

where $\overline{\tau}_\varepsilon$is given in (6.2.16) and $\hat{\tau}$ is the smallest time such that

$$C\,\left(1+\overline{R}_\varepsilon^2\right)^{\frac{1}{1-m}}\,\left[\mathcal{F}[u_0]\,e^{-4\hat{\tau}}\right]^{\frac{1-\vartheta}{2}} < \varepsilon\,,$$

where C and $\overline{R}_\varepsilon$ are given in (6.2.17) and in (6.2.26) respectively. In what follows we want to estimate $T_\star^\varepsilon(u_0)$ with $\varepsilon = \overline{\varepsilon}$.

Let us estimate $\overline{\tau}_\varepsilon$ with $\varepsilon = \overline{\varepsilon}$, from (6.2.16) we learn that (roughly speaking) $\overline{\tau}_\varepsilon$ is the maximum of the parameters t_0, t_1 and $\overline{t}$, given respectively in (6.2.3), (6.2.7) while $\overline{t} = c_{m,d}\,t_0$. The time t_1 depends only on d, m while t_0 also on A; it is anyway clear that under the running assumptions we can deduce the following bound

$$\overline{\tau}_{\hat{\varepsilon}} \le \tau_A$$

where $\tau_A = \tau_A(m, d, \overline{\varepsilon}, A) > 0$ is a uniform constant which depends only on m, d and the parameters $\overline{\varepsilon}$ and A.

Let us now estimate $\hat{\tau}$ with $\varepsilon = \overline{\varepsilon}$, from their expression it is clear that both ϑ and C depend only on m and d, while $\overline{R}_\varepsilon$ depends also on $\varepsilon > 0$. We recall that, under the running assumption, we have that $\mathcal{F}[u_0] \le B$ so if we define τ_B as the smallest time such that

$$C\,\left(1+\overline{R}_\varepsilon^2\right)^{\frac{1}{1-m}}\,\left[B\,e^{-4\tau_B}\right]^{\frac{1-\vartheta}{2}} < \varepsilon\,,$$

we will have that

$$\hat{\tau} \le \tau_B$$

and τ_B depends only on $m, d, \overline{\varepsilon}$ and B. We therefore conclude that

$$T_\star^{\overline{\varepsilon}}(u_0) \le \max\left\{\hat{\tau}, \overline{\tau}_{\overline{\varepsilon}}\right\} \le \max\left\{\tau_A\,, \tau_B\right\} =: \overline{N}\,,$$

which proves that $u_0 \in \mathcal{V}_{\overline{\varepsilon}}^{\overline{N}}$ as desired. The proof is concluded. $\square$

6.3.4 Proof of Corollary 6.1.3

As we already explained in Section 6.1.1, we can restate the statement of Corollary 6.1.3 in a language easier to deal with when we use the fast diffusion flow. Indeed, Corollary 6.1.3 is equivalent to the following Claim.

Claim. Under the hypothesis of Theorem 6.1.4 and assuming furthermore that

$$\int_{\mathbb{R}^d} u_0|x|^2 dx = \int_{\mathbb{R}^d} |x|^2\,\mathcal{B} dx\,, \qquad \text{and} \quad \delta[u_0] \le 1\,,$$

there exists a constant $\mathcal{C}_2 > 0$ such that

$$\delta[u_0] = \mathcal{I}[u_0] - 4\mathcal{F}[u_0] \geq \mathcal{C}_2 \frac{\left(\|\nabla u_0^{m-\frac{1}{2}} - \nabla \mathcal{B}^{m-\frac{1}{2}}\|_{\mathrm{L}^2(\mathbb{R}^d)}\right)^8}{\left(1 + \left\|\nabla u_0^{m-\frac{1}{2}}\right\|\right)^4} . \tag{6.3.18}$$

Proof of the claim We split the proof in different steps.

Step 1. In this step we prove that, under the assumption of Theorem 6.1.4 the following inequality holds

$$\delta[u_0] = \mathcal{I}[u_0] - 4\mathcal{F}[u_0] \geq \frac{\mathcal{C}_1}{\mathcal{C}_1 + 4} \mathcal{I}[u_0] . \tag{6.3.19}$$

Inequality (6.3.19) can be deduced from inequality (6.1.12). Indeed, multiplying inequality (6.1.12) by $4/(4+\mathcal{C}_1)$ we are left with

$$\frac{4}{4+\mathcal{C}_1}\mathcal{I}[u_0] - 4\mathcal{F} \geq 0 ,$$

from which inequality (6.3.19) can be deduced with no difficulties.

Step 2. In this step we want to prove that there exist a (computable) constant $\mathcal{C} > 0$ such that

$$\frac{4\,(m-1)^2}{(2m-1)^2}\left\|\nabla u_0^{m-\frac{1}{2}} - \nabla \mathcal{B}^{m-\frac{1}{2}}\right\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \leq \mathcal{C}\left(1 + \|\nabla u_0^{m-\frac{1}{2}}\|_{\mathrm{L}^2(\mathbb{R}^d)}\right) \max\{\mathcal{I}[u_0]^{\frac{3}{4}}, 1\}\, \mathcal{I}[u_0]^{\frac{1}{4}} . \tag{6.3.20}$$

Once inequality (6.3.20) is proven it is not difficult to deduce (6.3.18) using inequality (6.3.19) in the above formula.

Proof of Step 2. Recall that

$$\mathcal{B} = \left(1 + |x|^2\right)^{\frac{1}{m-1}}$$

and consider the following identities

$$\begin{aligned} \left|\nabla u_0^{m-\frac{1}{2}} - \nabla \mathcal{B}^{m-\frac{1}{2}}\right|^2 &= \left|\nabla u_0^{m-\frac{1}{2}}\right|^2 + \left|\nabla \mathcal{B}^{m-\frac{1}{2}}\right|^2 - 2\nabla u_0^{m-\frac{1}{2}} \cdot \nabla \mathcal{B}^{m-\frac{1}{2}} , \\ u_0\left|\nabla u_0^{m-1} - \nabla \mathcal{B}^{m-1}\right|^2 &= u_0\left|\nabla u_0^{m-1}\right|^2 + u_0\left|\nabla \mathcal{B}^{m-1}\right|^2 - 2u_0 \nabla u_0^{m-1} \cdot \nabla \mathcal{B}^{m-1} . \end{aligned} \tag{6.3.21}$$

Taking into account the following computations

$$\nabla u_0^{m-\frac{1}{2}} = \frac{\left(m-\frac{1}{2}\right)}{(m-1)} u_0^{\frac{1}{2}} \nabla u_0^{m-1} \qquad \text{and} \qquad \nabla u_0^{m-1} = (m-1)\, u_0^{m-2} \nabla u_0 , \tag{6.3.22}$$

the following holds

$$\begin{aligned} &\frac{4\,(m-1)^2}{(2m-1)^2}\left\|\nabla u_0^{m-\frac{1}{2}} - \nabla \mathcal{B}^{m-\frac{1}{2}}\right\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathcal{I}[u_0] = \\ &= \int_{\mathbb{R}^d} (\mathcal{B} - u_0)|\nabla \mathcal{B}^{m-1}|^2 dx + 2\int_{\mathbb{R}^d} u_0^{\frac{1}{2}}\left(u_0^{\frac{1}{2}} - \mathcal{B}^{\frac{1}{2}}\right) \nabla u_0^{m-1} \cdot \nabla \mathcal{B}^{m-1} dx . \end{aligned} \tag{6.3.23}$$

Since $\nabla \mathcal{B}^{m-1} = 2x$ and since we have chosen $\int_{\mathbb{R}^d} u_0 |x|^2 dx = \int_{\mathbb{R}^d} \mathcal{B}|x|^2 dx$ we have that (6.3.23) becomes

$$\frac{4\,(m-1)^2}{(2m-1)^2}\left\|\nabla u_0^{m-\frac{1}{2}} - \nabla \mathcal{B}^{m-\frac{1}{2}}\right\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathcal{I}[u_0] = 4\int_{\mathbb{R}^d} u_0^{\frac{1}{2}}\left(u_0^{\frac{1}{2}} - \mathcal{B}^{\frac{1}{2}}\right) \nabla u_0^{m-1} \cdot x \; dx . \tag{6.3.24}$$

Using again (6.3.22) we find that

$$\frac{4\,(m-1)^2}{(2m-1)^2}\left\|\nabla u_0^{m-\frac{1}{2}}-\nabla\mathcal{B}^{m-\frac{1}{2}}\right\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\mathcal{I}[u_0]=4\frac{2(m-1)}{2m-1}\int_{\mathbb{R}^d}\left(u_0^{\frac{1}{2}}-\mathcal{B}^{\frac{1}{2}}\right)\nabla u_0^{m-\frac{1}{2}}\cdot x\,dx\,.$$

Finally, by Cauchy-Schwartz we find that

$$\begin{aligned}&\frac{4\,(m-1)^2}{(2m-1)^2}\left\|\nabla u_0^{m-\frac{1}{2}}-\nabla\mathcal{B}^{m-\frac{1}{2}}\right\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\\&\le\mathcal{I}[u_0]+8\frac{(1-m)}{(1-2m)}\,\|\nabla u_0^{m-\frac{1}{2}}\|_{\mathrm{L}^2(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}\left(u_0^{\frac{1}{2}}-\mathcal{B}^{\frac{1}{2}}\right)^2|x|^2dx\right)^{\frac{1}{2}}\,.\end{aligned}\tag{6.3.25}$$

Let us define $w=u_0/\mathcal{B}$, we have then

$$\left(u_0^{\frac{1}{2}}-\mathcal{B}^{\frac{1}{2}}\right)^2|x|^2=\mathcal{B}\left(w^{\frac{1}{2}}-1\right)^2|x|^2=\mathcal{B}^m\left(w^{\frac{1}{2}}-1\right)^2\frac{|x|^2}{1+|x|^2}\,,$$

where in the last equality we have used the fact that $\mathcal{B}^{1-m}=\left(1+|x|^2\right)^{-1}$. It is not hard to verify that for $w\ge 0$ we have

$$|w^{\frac{1}{2}}-1|^2\le|w-1|\,,$$

and therefore the last integral in (6.3.25) can be estimated as

$$\int_{\mathbb{R}^d}\left(u_0^{\frac{1}{2}}-\mathcal{B}^{\frac{1}{2}}\right)^2|x|^2dx\le\int_{\mathbb{R}^d}|w-1|\mathcal{B}^m dx\,.$$

We learn from [17, Theorem 4] that there exists a computable constant $c_{m,d}>0$ such that

$$\left(\int_{\mathbb{R}^d}|w-1|\,\mathcal{B}^m dx\right)^2\le c_{m,d}\mathcal{F}[u_0]\,.$$

Therefore, we deduce from (6.3.25) that

$$\begin{aligned}\frac{4\,(m-1)^2}{(2m-1)^2}\left\|\nabla u_0^{m-\frac{1}{2}}-\nabla\mathcal{B}^{m-\frac{1}{2}}\right\|^2_{\mathrm{L}^2(\mathbb{R}^d)}&\le\mathcal{I}[u_0]+8c_{m,d}\frac{(1-m)}{(1-2m)}\,\|\nabla u_0^{m-\frac{1}{2}}\|_{\mathrm{L}^2(\mathbb{R}^d)}\,\mathcal{F}[u_0]^{\frac{1}{4}}\\&\le\mathcal{C}\left(1+\|\nabla u_0^{m-\frac{1}{2}}\|_{\mathrm{L}^2(\mathbb{R}^d)}\right)\max\{\mathcal{I}[u_0]^{\frac{3}{4}},1\}\,\mathcal{I}[u_0]^{\frac{1}{4}}\,.\end{aligned}\tag{6.3.26}$$

The proof is concluded. □

Appendix

6.4 Interpolating between L^p and C^ν norms – A lemma by Gagliardo and Nirenberg

The purpose of this note is to prove the following interpolation lemma which goes back to [141] and to [142, pag. 126]. In those papers the authors provide a more general result, here we prove only a simpler case which is needed to our purposes.

Lemma 6. *Let $p \geq 1$, $\nu \in (0,1)$ and $u : \mathbb{R}^d \to \mathbb{R}$ be a function such that $u \in L^p(\mathbb{R}^d) \cap C^\nu(\mathbb{R}^d)$. Then there exists a positive constant $C_{d,\nu,p}$ such that*

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq C_{d,\nu,p} \lfloor u \rfloor_{C^\nu(\mathbb{R}^d)}^{\frac{d}{d+p\nu}} \|u\|_{L^p(\mathbb{R}^d}^{\frac{p\nu}{d+p\nu}}, \tag{6.4.1}$$

where

$$C_{d,\nu,p} = \left(2^{p+1}\frac{\omega_d + 1}{\omega_d}\right)^{\frac{1}{p}} \left[\left(\frac{d}{p\nu}\right)^{\frac{p\nu}{p\nu+d}} + \left(\frac{p\nu}{d}\right)^{\frac{d}{p\nu+d}}\right]^{\frac{1}{p}},$$

where $\lfloor\cdot\rfloor_{C^\nu(\mathbb{R}^d)}$ is defined in (6.2.19) *and $\omega_d = |\mathbb{S}^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$.*

Proof. By the triangle inequality for any $x, y \in \mathbb{R}^d$ we have that

$$|u(x)|^p \leq (|u(x) - u(y)| + |u(y)|)^p \leq 2^p \left(|u(x) - u(y)|^p + |u(y)|^p\right).$$

Averaging on a ball $B_R(x)$ (for any $R > 0$) we obtain

$$\begin{aligned}
|u(x)|^p &\leq \frac{2^p}{\omega_d R^d}\int_{B_R(x)} |u(x)-u(y)|^p dy + \frac{2^p}{\omega_d R^d}\int_{B_R(x)} |u(y)|^p dy \\
&\leq \frac{2^{p+1}}{\omega_d}\left[R^{-d}\int_{B_R(x)} |u(x)-u(y)|^p dy + R^{-d}\int_{B_R(x)} |u(y)|^p dy\right] \\
&\leq \frac{2^{p+1}}{\omega_d}\left[R^{p\nu-d}\int_{B_R(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{p\nu}} dy + R^{-d}\|u\|^p_{L^p(\mathbb{R}^d)}\right] \\
&\leq 2^{p+1}\frac{\omega_d+1}{\omega_d}\left[R^{p\nu}\lfloor u\rfloor^p_{C^\nu(\mathbb{R}^d)} + R^{-d}\|u\|^p_{L^p(\mathbb{R}^d)}\right]
\end{aligned}$$

where $\omega_d = |S^{d-1}|$; in the third step we have used definition (6.2.19) and that $\int_{B_R(x)} |u(y)|^p dy \leq \|u\|^p_{L^p(\mathbb{R}^d)}$. Optimizing in R and then raising both members to the power $1/p$, proves inequality (6.4.1). □

6.5 Behaviour of the relative entropy when $v(\tau, y)$ approaches the Barenblatt profile $\mathcal{B}$

In [128] the authors noticed that the relative entropy $\mathcal{F}[v]$ behaves as a weighted L^2 the solution $v(\tau, y)$ approaches the Barenblatt profile $\mathcal{B}$. We report here their result which is used in the proof of Theorem 6.2.9, we refer to [128, Lemma 3] for a proof if this result.

Lemma 7. *Assume that $m \in (m_1, 1)$ and $0 < \varepsilon < 1$ are given. If v is a solution of* (6.1.14) *satisfying*

$$(1-\varepsilon) \leq \frac{v(\tau, y)}{\mathcal{B}(y)} \leq (1+\varepsilon) \quad \text{for any} \quad \tau \geq \underline{\tau}$$

then for any $\tau \geq \underline{\tau}$ we have

$$(1+\varepsilon)^{m-2} \int_{\mathbb{R}^d} \left| \frac{v(\tau, y)}{\mathcal{B}(y)} - 1 \right|^2 \mathcal{B}_\star^m dy \leq \frac{2}{m} \mathcal{F}[v(\tau, \cdot)] \leq (1-\varepsilon)^{m-2} \int_{\mathbb{R}^d} \left| \frac{v(\tau, y)}{\mathcal{B}(y)} - 1 \right|^2 \mathcal{B}_\star^m dy \,. \quad (6.5.1)$$

6.6 Quantitative regularity result for linear parabolic equations with measurable coefficients

Consider a linear parabolic equation of the form

$$\partial_t v = \nabla \cdot (A(t, x) \nabla v) \quad (6.6.1)$$

where $A(t, x)$ is a real symmetric matrix with bounded measurable coefficients satisfying the uniform parabolicity condition

$$0 < \lambda_0 |\xi|^2 \leq \sum_{i,j=1}^{d} A_{i,j}(t, x) \xi_i \xi_j \leq \lambda_1 |\xi|^2$$

for some $0 < \lambda_0 < \lambda_1$. In what follows it is convenient to use the notion of *parabolic distance* among two sets of the form $Q_1 = (T_1, S_1) \times \Omega_1$ and $Q_2 = (T_2, S_2) \times \Omega_2$, assuming that $Q_1 \subset Q_2$ we define

$$d_p(Q_1, Q_2) := \inf_{\substack{(t,x) \in \{[T_2, S_2] \times \partial\Omega_2\} \cup \{\{T_2\} \times \Omega_2\}, \\ (s,y) \in Q_1}} |x - y| \vee |t - s|^{1/2} \,. \quad (6.6.2)$$

We are now able to state a result of Hölder continuity for bounded local weak solutions to (6.6.1), for a definition of such concept see [154, Chp. 3]. Similar results are contained in [74, 154, 139], however the following one has the very useful feature that the dependence on the parameters λ_0, λ_1 is explicit.

Proposition 6.6.1. *Let Q_1 and Q_2 as above and assume that $2D = d_p(Q_1, Q_2)$. Then there exist a positive constant $\nu \in (0, 1)$ such that if v is a nonnegative bounded local weak solution on Q_2 then*

$$\sup_{(t,x), (\tau, y) \in Q_1} \frac{|v(t, x) - v(\tau, y)|}{\left(|x - y| + |t - \tau|^{1/2}\right)^\nu} \leq \frac{8^\nu}{D^\nu} \|v\|_{L^\infty(Q_2)} \,, \quad (6.6.3)$$

where ν depends only on d, λ_0, λ_1 and it is given

$$\nu := \log_8 \frac{\kappa_\ell^{\lambda_0^{-1} + \lambda_1}}{\kappa_\ell^{\lambda_0^{-1} + \lambda_1} - 1} \in (0, 1) \,, \quad (6.6.4)$$

and $\kappa_\ell \geq 2$ depends only on the dimension d.

The proof of the previous result can be found in [139, Prop. 4.2].

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